On Sequential and Simultaneous Contributions under Incomplete Information*

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Abstract

When contributors to a common cause (or, public good) are uncertain about each others' valuations, early contributors are likely to be cautious in free-riding on future contributors. Contrary to the case of complete information, when contributors have independent private valuations for the public good, the expected total contribution generated in a sequential move game *may* be higher than in a simultaneous move game. This is established in a conventional framework with quasi-linear utility where agents care only about the total provision of the public good (rather than individual contribution levels) and there is *no* non-convexity in the provision of the public good. We allow for arbitrary number of agents and fairly general distribution of types.

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1 Introduction

Economic agents often contribute sequentially towards a public good or a project of common interest. In many cases, the order in which the agents contribute is determined *exogenously.* National governments may commit resources and defence forces to war efforts in a hierarchical order depending on individual stakes in the conflict. R&D investment by private firms with shared research output (as in open source software development or research on mapping of genomes) sometimes follows the scientific order of discoveries with firms specializing in "basic" research moving earlier (e.g., Bessen and Maskin, 2006).[1] In team production, participants can be deliberately induced to exert efforts in an exogenous sequential order under full observability of efforts (Winter, 2006). Political lobbying or contributions to reduction of global environmental damage by countries may follow a specific order according to perceived leader-follower roles. Finally, contributions to charities by donors during fundraising may follow the order in which the donors are contacted.

In many of the examples discussed above, early contributions are observed by, or revealed to, other contributors before they make their own contributions. Governments announce their own commitments to war efforts or disaster relief. In independent R&D ventures with shared output, leading firms may publicly highlight their investment (or make their research output public) so that the follower firms make investment decisions based on this knowledge. During fundraising campaigns, charities announce donations as they come in.

When contributions are not observed till the contribution process has ended, the strategic interaction between potential contributors is identical to that in a simultaneous move game. On the other hand, the ability to observe actions taken by prior contributors generates a sequential game of voluntary contributions. A natural question that arises in this context is the effect of observability of contributions on the strategic incentives of voluntary contributors and on the eventual provision of the public good.

Varian (1994) has argued that in sequential games of voluntary contribution to a pure public good where the order of moves is exogenous, the ability of late movers to observe the contributions made by early movers aggravates the free rider problem. This is particularly striking if each agent has only one chance to con-

[1]Bessen and Maskin analyze the merits of sequential discoveries with public good features in software developments (due to complementarity between successive innovations).

tribute and cannot add to her current contribution later (i.e., early movers can credibly pre-commit to a certain level of contributions); in such situations, the total contribution generated is never greater and can be significantly smaller relative to the game where contributions are not revealed or observed (i.e., a simultaneous contribution game). An important assumption behind Varian's result is *complete information* – agents know each others' valuations of the public good. Intuition suggests that if agents do not know each others' valuations (the contribution game is one of incomplete information), then early movers who commit to low contributions (in order to free ride on late movers) also face the risk that late movers may not value the public good as much and thus, under-provide it relative to what the early movers may consider acceptable. Furthermore, when agents cannot observe others' contributions (as in a simultaneous move game), a contributor with fairly low valuation has an incentive to contribute very little as she gambles on the event that other contributors have much higher valuations and will contribute generously; this, in turn, means that in states of the world where a large proportion of contributors are actually ones with very low valuations, the total contribution in a simultaneous move game is excessively small relative to what these low valuation contributors would have provided under complete information. When contributions are sequentially observed this problem is partially redressed, because later contributors make their donations *knowing* what the earlier ones have contributed and do not need to guess their contributions.

This paper develops the above intuition to show that under incomplete information about individual (independent) private valuations for the public good, a sequential contribution game may actually generate higher total expected contribution than the simultaneous move contribution game.[2] We establish this in a conventional economic framework (as in Varian, 1994) where contributors care only about the total provision of the public good (rather than individual contribution levels of other agents), thus ruling out snob effects, warm glow and more generally, complementarities between individual contributions. This distinguishes our work from Romano and Yildirim (2001) who show that sequential contribution may increase total contributions if utility depends not just on total contributions but also

[2]In a brief section, Varian (1994) discusses the incomplete information game considered by us and argues that incomplete information about the second-mover's utility leads the first-mover to contribute *less than what she would under complete information.* He does not compare sequential and simultaneous contributions under incomplete information.

on individual contribution levels. Further, there is no non-convexity in the production technology. This rules out the presence of increasing returns or threshold effect in the production technology; earlier, Andreoni (1998) had shown that in the presence of such effects, learning about increased contribution of early movers may increase the marginal productivity of followers' contributions, thus creating advantages for a sequential contribution format.[3,4]

We assume that agents have quasi-linear utility and allow for arbitrary number of agents and fairly general distribution of types. We show that, under certain sufficient conditions that include *concavity of the marginal utility* from the public good, the expected total contribution generated in a perfect Bayesian equilibrium of a sequential move contribution game is at least as large as that in a Bayes-Nash equilibrium of the simultaneous move contribution game if the agent who moves last in the sequential game is one who makes a strictly positive contribution in the simultaneous move game for every possible realization of her type. We obtain this result, even though each agent has only one chance to contribute and can therefore pre-commit to contributions;[5] note that under complete information, early movers have the greatest ability to free ride on late movers precisely when agents have such commitment ability. Further, if the equilibrium in the sequential game is one where some contributor who moves prior to the last mover, makes a strictly positive

[3]One may view discrete public goods (such as a 0-1 public good that is either provided or not provided) as special cases of such threshold technology. Admati and Perry (1991) and Marx and Matthews (2000) respectively analyze the provision of discrete public goods (under complete information) in an alternating offer voluntary contributions format and an unrestricted repeated contributions format. Menezes et al. (2001), and Agastya et al. (2007) analyze simultaneous contribution to discrete public goods under incomplete information about private valuations.

[4]Vesterlund (2003) considers a common-value public good model with uncertainty about the *common value*. By announcing contributions and inducing a sequential move game, a high-quality charity is able to raise more funds. Ours is an independent private-values model. Other explanations of contribution announcement by charities such as wealth signaling and prestige motives (Glazer and Konrad, 1996; Harbaugh, 1998) do not explain why announcements are made during the contribution process rather than at the end of it.

[5]Winter (2006) examines incentive design in team production problem under complete information with an exogenously given sequential order of tasks where the tasks performed by agents are perfectly complementary. In our model, the participants' contributions are perfect substitutes (rather than complements) and there are no direct, differential incentives for participants except that all participants get equal access to the total public good produced. Under incomplete information, our intuition favoring sequential contributions may carry over to some team production technologies with less than perfect complementarity.

contribution with strictly positive probability or alternatively, if the marginal utility from the public good is *strictly* concave, then the sequential game generates strictly higher expected total contribution. In an example, with two-agents and two-types, we show that some of these sufficient conditions are not necessary for the sequential game to generate higher contribution.

The sequential move game considered in this paper is one where every agent perfectly observes the individual contributions made by earlier contributors before making her own contribution decision. However, as we argue informally at the end of section 4, our results are equally valid for a modified version of the sequential move game where each contributor observes only the *sum of contributions* made by earlier contributors, but not necessarily their individual contributions.

In an earlier paper (Bag and Roy, 2008), we consider a multistage game of contribution to a public good with *all agents having the option to contribute in <u>all</u> stages*. There we show that under incomplete information about the agents' (independent private) valuations of the public good, the expected total contribution is higher if the contributions made at each stage are observed before the next stage; the economic reasoning behind this is based on the possibility of revelation of the private preferences of contributors through their actions and the incentive of higher valuation contributors to hide information about their true preferences as it may make them more vulnerable to free-riding by other agents in subsequent stages of the game. Such incentives are not important in voluntary contribution processes where agents cannot contribute repeatedly; the current paper focuses on these environments by assuming that agents move sequentially with each agent having only one turn to contribute so that later movers cannot free ride on earlier contributors, and contributors have no incentive to either hide or reveal their private valuation of the public good.

There is a substantial literature on voluntary provision of public goods under incomplete information that focus on *inefficiencies* that arise due to incompleteness of information.[6] We do not concern with the efficiency or normative issues; instead, we offer an incomplete information based explanation of why sequential contribution schemes may be better for the total provision of the public good.

The next section presents the model. In section 3 we analyze the simultaneous contribution game, followed by an analysis of the sequential contribution game in

[6] See Bliss and Nalebuff (1984), Palfrey and Rosenthal (1988), Fershtman and Nitzan (1991), Gradstein (1992), Vega-Redondo (1995), etc.

section 4. In section 5, we state our main results comparing the expected contributions in the sequential and simultaneous move games and in particular, outline general conditions under which the sequential form generates higher expected total contribution. Section 6 elaborates further on these conditions. Section 7 discusses the two-agents, two-types case. The appendix contains some proofs not included in the main text.

2 The model

$N > 1$ agents contribute voluntarily to a public good. Each agent $i \in \{1, ..., N\}$ has a budget constraint $w_i > 0$. Agent i's payoff depends on the total contribution of all agents, her own contribution and her own type, and is given by

$$u_i(g_i, g_{-i}, \tau_i) = \tau_i V_i(g_i + g_{-i}) + w_i - g_i,$$

where $g_i \geq 0$ is i's contribution, g_{-i} is the total contribution of all other agents $j \neq i$, and τ_i is a private preference parameter of agent i that affects her marginal utility from consumption of the public good, is known only to agent i and is interpreted as the "type" of agent i. It is common knowledge that each agent i's type τ_i is an independent random draw from a probability distribution with distribution function F_i and *compact* support $A_i \subset \mathcal{R}_{++}$. Let $\underline{\tau}_i$ and $\overline{\tau}_i$ be the lowest and highest possible types of agent i defined by

$$\underline{\tau}_i = \min\{\tau : \tau \in A_i\}, \quad \overline{\tau}_i = \max\{\tau : \tau \in A_i\}.$$

Assumption 1. $\forall i \in \{1, ..., N\}$, $V_i(.)$ *is continuously differentiable, concave and non-decreasing on* $\mathcal{R}_+$, *with* $V_i(0) = 0$, $\underline{\tau}_i V_i'(0) > 1$ *and* $\overline{\tau}_i V_i'(w_i) < 1$.

Define $\forall i$:

$$z_i = \sup\{x \geq 0 : \ \overline{\tau}_i V_i'(x) = 1\}.$$

Under Assumption 1, $0 < z_i < w_i$. It is easy to check that agent i would never contribute in excess of z_i in any contribution game, whatever be her type. This allows us to drop w_i and write agent i's payoff function simply as

$$u_i = \tau_i V_i(g_i + g_{-i}) - g_i.$$

Define

$$\overline{G} = \sum_{i=1}^{N} z_i > 0.$$

Assumption 2. $V_i(.)$ *is strictly concave on* $[0, \overline{G}]$, $\forall i \in \{1, ..., N\}$.

Under Assumptions 1 and 2, every agent i of every possible type $\tau_i \in A_i$ has a unique *standalone* contribution $x_i(\tau_i) \in (0, w_i)$ defined by:

$$\begin{aligned} x_i(\tau_i) &= \arg\max_{g_i} u_i(g_i, 0, \tau_i), \\ \text{satisfying} \qquad \tau_i V_i'(x_i(\tau_i)) &= 1. \end{aligned} \tag{1}$$

It is easy to check that $x_i(\tau_i)$ is strictly increasing in τ_i, implying:

$$z_i = x_i(\overline{\tau}_i).$$

The *expected* standalone contribution of agent i, denoted hereafter by θ_i, is given by:

$$\theta_i = \int_{A_i} x_i(\tau_i)\, \mathrm{d}F_i(\tau_i).$$

Also, since $V_i(.)$ is non-decreasing, $V_i'(.) \geq 0$. Assumption 2 therefore implies that $V_i'(G) > 0$ on $[0, \overline{G})$. Finally, we impose:

Assumption 3. $V_i'(.)$ *is concave on* $[0, \overline{G}]$.

Assumption 3 is an important technical restriction that will be useful in comparing expected total contributions under sequential and simultaneous move games.[7] Note that we do not require that the function $V_i'(.)$ be concave or strictly decreasing on the entire positive real line.

While our analysis is presented for a continuously variable public good, by setting $V_i(G) = V(G)$ for all i and interpreting $V(G)$ as the *'probability of success'* of a public project with binary outcomes ("success" or "failure") that depends on total investment G, the analysis can be easily applied to a discrete public good

[7] A simple example that satisfies all of the above assumptions is the situation where all agents have identical utility functions and distribution of types and for all $i = 1, ..., N$, $V_i(.)$ is the quadratic function

$$V_i(G) = \begin{cases} \alpha G - \frac{1}{2}G^2, & 0 \leq G \leq \alpha,\ \alpha > 0 \\ \frac{1}{2}\alpha^2, & G > \alpha, \end{cases}$$

with the additional restrictions that $\underline{\tau}_i = \underline{\tau}$, $\overline{\tau}_i = \overline{\tau}$ satisfy

$$\frac{1}{\alpha \underline{\tau}} < 1, \quad \frac{1}{\alpha \overline{\tau}} \geq 1 - \frac{1}{N}.$$

setting;[8] in that case, $\tau_i > 0$ is agent i's deterministic utility if the project succeeds, while the utility obtained when the project fails is normalized to zero.

We will compare two game forms – an N-stage *sequential contribution game* and a *simultaneous contribution game* – and the solution concepts are *Perfect Bayesian Equilibrium* and *Bayesian-Nash Equilibrium*, respectively. In the sequential contribution game, the agents contribute in an exogenous order and the contribution amounts become known as and when they are made. Each agent is allowed to contribute only once and is not allowed to add to her contribution at a later stage. In the simultaneous contribution game, each agent contributes without any knowledge of other agents' contributions. We confine attention to pure strategy equilibria.

The contribution games are compared according to the (*ex ante*) expected total contributions made by all N agents i.e., the expected provision of public good.

3 Simultaneous contribution game

First, we analyze the simultaneous contribution game. Let $y_i(\tau_i)$ denote the equilibrium contribution of agent i of type $\tau_i \in A_i$, $i \in \{1, ..., N\}$ and $y_{-i}(\tau_{-i}) = \sum_{j \neq i} y_j(\tau_j)$ where $\tau_{-i} \in \Pi_{j \neq i} A_j$ is the vector of types for agents other than agent i. Then, $y_i(\tau_i)$ is a solution to the following expected utility maximization problem:

$$\max_{y \geq 0} \quad \tau_i E_{\tau_{-i}}[V_i(y + y_{-i}(\tau_{-i}))] - y. \tag{2}$$

In what follows, we denote by $F_{-i}(\tau_{-i})$ the joint distribution of τ_{-i}. We start with a simple observation that follows directly from the definitions of $x_i(\tau_i)$ and $\overline{G}$:

Lemma 1. *Consider any Bayesian-Nash equilibrium of the simultaneous move contribution game where $y_i(\tau_i)$ is the contribution made by agent i of type τ_i. Then, $y_i(\tau_i) \leq x_i(\tau_i)$ $\forall \tau_i \in A_i$, $\forall i \in \{1, ..., N\}$, and the total contributions generated*

$$\sum_{i=1}^{N} y_i(\tau_i) \leq \overline{G} \quad \textit{with probability one.}$$

Below we derive a much sharper bound for the *expected* total contribution, but first the following technical result should be noted (see the Appendix for the proof).

[8] Consider any density function $h(z)$ with support on the positive real line that is weakly decreasing and is, in addition, strictly decreasing and concave over $[0, \overline{G}]$. Then, taking $V(x) = \int_0^x h(z)\, dz$ as the probability of success of the project satisfies our assumptions.

Lemma 2. *Consider the simultaneous move game and fix an agent index i. If for some $j \neq i$, $Pr\{\tau_j : y_j(\tau_j) > 0\} > 0$, then the probability distribution of the random variable $y_{-i} = \sum_{j \neq i} y_j$ is non-degenerate.*

Lemma 3. (***An upper-bound on expected total contribution***) *Consider any Bayesian-Nash equilibrium of the simultaneous move contribution game where for some agent i, $y_i(\tau_i)$, the equilibrium contribution of agent i, is strictly positive τ_i-almost surely i.e.,* $\Pr\{\tau_i : y_i(\tau_i) > 0\} = 1$.

(i) Then,

$$\sum_{k=1}^{N} \left\{ \int y_k(\tau_k)\, \mathrm{d}F_k(\tau_k) \right\} \leq \theta_i \tag{3}$$

i.e., the expected total contribution by all agents generated in this equilibrium does not exceed the expected standalone contribution of agent i.

(ii) If, in addition, $V_i'(G)$ is strictly concave on $[0, \overline{G}]$ and there exists some agent $j \neq i$ such that $\Pr\{\tau_j : y_j(\tau_j) > 0\} > 0$, *then the inequality in (3) holds strictly i.e., the expected total contribution by all agents generated in this equilibrium is strictly less than the expected standalone contribution of agent i.*

Proof. Using the first-order condition of the maximization problem (2) faced by agent i of type τ_i and the hypothesis that $y_i(\tau_i) > 0$ τ_i-almost surely, we have:

$$\begin{aligned} & \tau_i \int V_i'(y_i(\tau_i) + y_{-i}(\tau_{-i}))\, \mathrm{d}F_{-i}(\tau_{-i}) \\ = \;& \tau_i E_{\tau_{-i}}[V_i'(y_i(\tau_i) + y_{-i}(\tau_{-i}))] \\ = \;& 1, \quad \tau_i\text{-almost surely,} \end{aligned} \tag{4}$$

First, we establish (i). Since (using Assumption 3) $V_i'(.)$ is concave on $[0, \overline{G}]$ and from Lemma 1, $y_i(\tau_i) + y_{-i}(\tau_{-i}) \in [0, \overline{G}]$ almost surely, we have by Jensen's inequality

$$\tau_i V_i'(y_i(\tau_i) + E_{\tau_{-i}}\{y_{-i}(\tau_{-i})\}) \geq 1, \quad \tau_i\text{-almost surely}$$

so that using (Assumption 2) concavity of $V_i(.)$ and (1) it follows that

$$y_i(\tau_i) + E_{\tau_{-i}}\{y_{-i}(\tau_{-i})\} \leq x_i(\tau_i), \quad \tau_i\text{-almost surely}$$

and integrating with respect to the distribution of agent i's type we have:

$$E_{\tau_i}\{y_i(\tau_i)\} + E_{\tau_{-i}}\{y_{-i}(\tau_{-i})\} \leq E_{\tau_i}\{x_i(\tau_i)\} = \theta_i,$$

establishing part (i) of the lemma.

Now, consider part (ii) of the lemma. It is easy to check from the first-order conditions of agent j's maximization problem that given $y_{-j}(\tau_{-j})$, the equilibrium contribution $y_j(\tau_j)$ of agent j is non-decreasing in the type τ_j of agent j. Thus, if $\tau < \tau'$, $\tau, \tau' \in A_j$ and $y_j(\tau) > 0$, then $y_j(\tau') > 0$. Further, using (4),

$$\begin{aligned} 1 &= \tau \int V_j'(y_j(\tau) + y_{-j}(\tau_{-j}))\, \mathrm{d}F_{-j}(\tau_{-j}) \\ &< \tau' \int V_j'(y_j(\tau) + y_{-j}(\tau_{-j}))\, \mathrm{d}F_{-j}(\tau_{-j}) \end{aligned}$$

so that $y_j(\tau') > y_j(\tau)$. Since $\Pr\{\tau_j : y_j(\tau_j) > 0\} > 0$ for some $j \neq i$, the total contribution of agents other than agent i, $y_{-i}(\tau_{-i})$, is a non-degenerate random variable (by Lemma 2). Therefore, using strict concavity of $V_i'(.)$ on $[0, \overline{G}]$ (assumed in part (ii) of the lemma), the fact that $y_j(\tau_j) + y_{-j}(\tau_{-j}) \in [0, \overline{G}]$ almost surely (Lemma 1), and Jensen's inequality, we have

$$\tau_i V_i'(y_i(\tau_i) + E_{\tau_{-i}}\{y_{-i}(\tau_{-i})\}) > 1, \quad \tau_i\text{-almost surely}$$

so that using strict concavity of $V_i(.)$ on $[0, \overline{G}]$ (Assumption 2) and (1) it follows that

$$y_i(\tau_i) + E_{\tau_{-i}}\{y_{-i}(\tau_{-i})\} < x_i(\tau_i), \quad \tau_i\text{-almost surely}$$

and integrating with respect to the distribution of agent i's type we have:

$$E_{\tau_i}\{y_i(\tau_i)\} + E_{\tau_{-i}}\{y_{-i}(\tau_{-i})\} < E_{\tau_i}\{x_i(\tau_i)\} = \theta_i,$$

establishing part (ii) of the lemma. **Q.E.D.**

The next result follows immediately from Lemma 3:

Corollary 1. *Suppose there is a Bayesian-Nash equilibrium of the simultaneous move contribution game where* <u>*for all*</u> $i \in \{1, ..., N\}$, $y_i(\tau_i)$, *the equilibrium contribution of agent* i *of type* τ_i, *satisfies* $y_i(\tau_i) > 0$, τ_i*-almost surely. Then, the expected total contribution in this equilibrium does not exceed* $\min\{\theta_i : i = 1, ..., N\}$. *If, further,* $V_i'(G)$ *is strictly concave on* $[0, \overline{G}]$ <u>*for all*</u> $i \in \{1, ..., N\}$, *then the expected total contribution in this equilibrium is strictly less than* $\min\{\theta_i : i = 1, ..., N\}$.

4 Sequential contribution game

In this section, we analyze the sequential contribution games where agents contribute in an exogenous order of moves with each agent contributing only once.

Let

$$P = \{p = (p_1, ..., p_N) : (p_1, ..., p_N) \text{ is a permutation of } (1, ..., N)\}.$$

For each $p = (p_1, ..., p_N) \in P$, we can define an N-stage sequential contribution game $\Gamma(p)$ where agent p_i contributes (only) in the i-th stage after observing contributions made in all previous stages $1, ..., i-1$.

We first specify a lower bound on the total contribution that depends on the last mover's type:

Lemma 4. *In any perfect Bayesian equilibrium of $\Gamma(p)$, for each possible realization τ of the type of the last mover p_N, the total contribution generated is at least as large as $x_{p_N}(\tau)$, her standalone contribution for type τ, and the* expected *total contribution generated in the game is at least as large as θ_{p_N}, the* expected *standalone contribution of agent p_N.*

Proof. The proof follows from the fact that if $z \geq 0$ is the total contribution of agents in the first $(N-1)$ stages, then in the last stage of the game, the unique optimal action of agent p_N of type τ is to contribute $\max\{0, x_{p_N}(\tau) - z\}$. **Q.E.D.**

Next, we argue that as long as the total contribution generated in the first $(N-1)$ stages is strictly positive with some probability, the expected total contribution generated in the sequential game is strictly higher than the expected standalone contribution of the last mover. The main argument here is that earlier contributors know that even if they contribute zero, the last mover will ensure that the total contribution is at least as large as her standalone contribution (depending on her realized true type). If the total contribution on the equilibrium path in the first $(N-1)$ stages is below the standalone level for the lowest type of the contributor in stage N, then the last contributor who contributes strictly positive amount (for some realization of her type) among the first $(N-1)$ movers will always be better off deviating and contributing zero with probability one. Therefore, on an equilibrium path where total contribution in first $(N-1)$ stages is strictly positive (with strictly positive probability), it must exceed the standalone level of the N-th contributor for the very low realizations of her type.

Lemma 5. *In any perfect Bayesian equilibrium of* $\Gamma(p)$ *where the total contribution generated in the first* $(N-1)$ *stages is strictly positive with strictly positive probability, the expected total contribution is strictly higher than* θ_{p_N}*, the expected standalone contribution of agent* p_N*.*

Proof. In view of Lemma 4, it is sufficient to show that for an event (i.e., a set of type profiles for N agents) of strictly positive probability measure, the generated total contributions must strictly exceed the standalone contributions of agent p_N (corresponding to her realized types in those type profiles).

For each realization of types of the first $(N-1)$ contributors $\omega \in \prod_{i=1}^{N-1} A_i$, let $z(\omega)$ be the total contribution generated in the first $(N-1)$ stages. Observe that the unique optimal action of agent p_N of type τ in stage N is to contribute $\max\{0, x_{p_N}(\tau) - z\}$, if the total contribution in the first $(N-1)$ stages is z. We claim that since $z(\omega) > 0$ with strictly positive probability, it must be the case that

$$z(\omega) > x_{p_N}(\underline{\tau}_{p_N}) \quad \text{with strictly positive probability.} \tag{5}$$

Suppose, to the contrary, that $z(\omega) \leq x_{p_N}(\underline{\tau}_{p_N})$ almost surely. Then, given the optimal strategy of agent p_N, the total contribution generated at the end of the game is exactly identical to that generated if every agent contributes zero with probability one in the first $(N-1)$ stages. In particular, let $k = \max\{1 \leq n \leq N-1 :$ agent p_n makes strictly positive contribution with strictly positive probability$\}$.
By definition, for all n lying strictly between k and N, no contribution occurs (almost surely) on the equilibrium path in stage n. Consider a unilateral deviation where agent p_k contributes zero almost surely and independent of history. The distribution of total contribution generated at the end of the game remains unchanged (as the last mover makes up the difference). Therefore, this deviation is strictly beneficial for agent p_{N-k}. This establishes (5). From (5), it follows that there exists $\epsilon > 0$ small enough such that

$$\Pr\{\omega \in \prod_{i=1}^{N-1} A_i : z(\omega) > x_{p_N}(\underline{\tau}_{p_N}) + \epsilon\} > 0. \tag{6}$$

Let $\widehat{\tau} > \underline{\tau}_{p_N}$ be defined by :

$$\widehat{\tau} V'_{p_N}(x_{p_N}(\underline{\tau}_{p_N}) + \epsilon) = 1.$$

Choose $\widetilde{\tau} \in (\underline{\tau}_{p_N}, \widehat{\tau})$. Since $\underline{\tau}_{p_N} = \min\{\tau : \tau \in A_{p_N}\}$ and A_{p_N} is the support of the probability distribution of τ_{p_N}, it follows that $F_{p_N}(\widetilde{\tau}) > 0$. Also note that

$$x_{p_N}(\tau) < x_{p_N}(\underline{\tau}_{p_N}) + \epsilon, \forall \tau \in A_{p_N} \cap [\underline{\tau}_{p_N}, \widetilde{\tau}]. \tag{7}$$

as $x_i(\tau_i)$ is strictly increasing in τ_i for all i. Let

$$B_1 = \{\omega \in \prod_{i=1}^{N-1} A_i : z(\omega) > x_{p_N}(\tau), \tau \in A_{p_N} \cap [\underline{\tau}_{p_N}, \widetilde{\tau}]\}.$$

Then, using (6),

$$\Pr(B_1) > 0.$$

Let B be the event:

$$B = \{(\tau_{p_1}, ..., \tau_{p_{N-1}}) \in B_1, \tau_{p_N} \leq \widetilde{\tau}\}.$$

Since $\Pr(B_1) > 0$ and $F_{p_N}(\widetilde{\tau}) > 0$, it follows that $\Pr(B) > 0$. Further, using (6) and (7), for realizations of type profiles (of all players) in the set B, the generated total contributions strictly exceed the standalone contributions of agent p_N. The proof is complete. **Q.E.D.**

Finally, note that the set of perfect Bayesian equilibrium outcomes of the sequential game $\Gamma(p)$ is identical to the set of perfect Bayesian equilibrium outcomes generated when the extensive form is modified so that in stage i, $i = 2, ..., N$, agent p_i observes perfectly *the sum of the actual contributions* made by agents $p_1, ..., p_{i-1}$, but not their individual contributions. Note that the payoff of each agent depends only on the total contribution generated at the end of the game (and not on individual contribution levels) and further, each agent contributes only once so that any information about the private preferences of an agent inferred from her individual contribution level is of no relevance to agents who move later. The strategy sets of players in the modified sequential game is a subset of that in $\Gamma(p)$. However, sequential rationality ensures that in any perfect Bayesian equilibrium of $\Gamma(p)$, at each stage of the game and for each realized history, the strategy of the player of any type who moves at that stage must specify an action that is also optimal if the player observes only the total contribution generated (at the end of the previous stage for that realized history). Therefore, for any perfect Bayesian equilibrium of $\Gamma(p)$, working backwards from the last stage and substituting strategies of players that possibly depend on individual past contributions by ones that depend only on the sum of previous contributions, one can construct a perfect Bayesian equilibrium of the modified sequential game that is outcome equivalent. Our results in the subsequent sections comparing the expected total contributions generated in the sequential and simultaneous move games are therefore equally valid for versions of the sequential move game where individual contributions are observed imperfectly but the cumulative contribution generated before each stage is observed perfectly.

5 Comparison of contributions

We now present the paper's main results comparing the expected total contributions generated in the simultaneous and sequential move contribution games.

First, we provide sufficient conditions under which sequential move games that satisfy a certain restriction on the order of moves (specifically, in terms of who moves in the last stage) generate weakly greater expected total contributions compared to the simultaneous move game.

Proposition 1. *Consider any Bayesian-Nash equilibrium $\mathcal{E}$ of the simultaneous move contribution game where some agent i makes a strictly positive contribution τ_i-almost surely. Then, every perfect Bayesian equilibrium of the N-stage sequential move game $\Gamma(p)$ where $p = (p_1, ..., p_N)$ and $p_N = i$, generates at least as much expected total contribution as in the Bayesian-Nash equilibrium $\mathcal{E}$ of the simultaneous move game.*

Proof. Follows immediately from Lemma 3 and Lemma 4. **Q.E.D.**

Next, we state sufficient conditions under which a sequential move game generates *strictly* higher expected total contributions compared to the simultaneous move game.

Proposition 2. *Consider any Bayesian-Nash equilibrium $\mathcal{E}$ of the simultaneous move contribution game where some agent i makes a strictly positive contribution τ_i-almost surely. Consider any perfect Bayesian equilibrium $\widehat{\mathcal{E}}$ of the sequential move contribution game $\Gamma(p)$ where $p = (p_1, ..., p_N)$ and $p_N = i$. Suppose, further, that* at least one *of the following hold:*

(a) (Imperfect Free-Riding) In equilibrium $\widehat{\mathcal{E}}$ of the sequential game $\Gamma(p)$, $\exists \ell \neq i$ (i.e., some agent who moves in one of the first $(N-1)$ stages) who contributes strictly positive amount with strictly positive probability;

(b) $V_i'(G)$ *is strictly concave on* $[0, \overline{G}]$ *and there exists some agent* $j \neq i$ *such that in the equilibrium* $\mathcal{E}$ *of the simultaneous move contribution game* $\Pr\{\tau_j : y_j(\tau_j) > 0\} > 0$.

Then, equilibrium $\widehat{\mathcal{E}}$ *of the sequential move contribution game* $\Gamma(p)$ *generates strictly higher expected total contribution than equilibrium* $\mathcal{E}$ *of the simultaneous move game.*

Proof. If (a) holds, then the result follows from Lemma 3(i) and Lemma 5 . If (b) holds, then the result follows from Lemma 3(ii) and Lemma 4. **Q.E.D.**

The conditions in Proposition 1 and Proposition 2 are strong sufficient conditions; in particular, the conclusions of Proposition 1 and Proposition 2 may hold even if the last mover in the sequential move game is one who contributes zero with positive probability in the simultaneous move game and in fact, even if all players contribute zero with positive probability in the simultaneous move game. We will illustrate these possibilities in examples in section 7.

The next result follows immediately from Proposition 1 and Corollary 1:

Corollary 2. *Consider any Bayesian-Nash equilibrium* $\mathcal{E}$ *of the simultaneous move contribution game where every agent makes strictly positive contribution almost surely. Then, for every* $p \in P$, *every perfect Bayesian equilibrium of the* N*-stage sequential move game* $\Gamma(p)$ *generates at least as much expected total contribution as in equilibrium* $\mathcal{E}$ *of the simultaneous move game.*

Corollary 2 clarifies that if the equilibrium of the simultaneous move game (to which one compares the outcomes of the sequential games) is an "interior equilibrium" where all agents of "almost" all types contribute strictly positive amounts, then there is no need to impose any restriction on the order of moves in the sequential game to ensure that it generates weakly higher expected total contributions.

Similarly, using Proposition 2 and Corollary 1, we have immediately:

Corollary 3. *Consider any Bayesian-Nash equilibrium* $\mathcal{E}$ *of the simultaneous move contribution game where every agent makes strictly positive contribution almost surely. Suppose, further, that at least one of the following holds:*

(a) $\forall p \in P$, *in every perfect Bayesian equilibrium of the* N*-stage sequential move game* $\Gamma(p)$, *some agent that moves in the first* $(N-1)$ *stages contributes strictly positive amount with strictly positive probability;*

(*b*) $V_i'(G)$ *is strictly concave on* $[0, \overline{G}]$, $\forall i \in \{1, ..., N\}$.

Then, for every $p \in P$*, every perfect Bayesian equilibrium of the* N*-stage sequential move game* $\Gamma(p)$ *generates strictly higher expected total contribution than equilibrium* $\mathcal{E}$ *of the simultaneous move game.*

Corollary 3 indicates that if the equilibrium of the simultaneous move game is an "interior equilibrium" then, under certain additional conditions, the sequential move game generates strictly higher expected total contributions independent of the order of moves.

6 Further conditions

In the previous section, we outlined some conditions under which the expected total voluntary contribution generated in a sequential move game exceeds (weakly or strictly) that generated in a simultaneous move game. One restriction that is required in all of the results outlined in the previous section is that in the Bayesian-Nash equilibrium of the simultaneous move game, a certain agent (in particular, the one that moves last in the sequential game) must contribute strictly positive amount for almost every possible realization of her type. Another condition that ensures that the sequential game generates *strictly* higher expected total contribution requires that in the perfect Bayesian equilibrium of the N-stage sequential game, free-riding is imperfect i.e., some agent moving in the first $(N-1)$ stages contributes strictly positive amount with strictly positive probability.

In this section, we outline some sufficient and verifiable conditions on the exogenous preferences and distributions of types that ensure that the equilibria of the relevant games satisfy the two properties noted above. It is worth emphasizing that the conditions to be stated are by no means tight. One should view them as merely illustrative of how the antecedents of the main propositions (and corollaries) in the previous section may hold and as indicating that, in particular, the propositions are not vacuous.

While the conditions can be easily written for the general version of the model outlined in Section 2, for ease of exposition we will assume that:

Assumption 4. $V_i(G) = V(G)$, $\forall i \in \{1, ..., N\}$.

Note that Assumption 4 allows for the possibility that agents differ in the distribution of types (i.e., the distribution function F_i need not be identical) so that some degree of asymmetry between players is allowed for. However, preferences now only depend on the type of the agent and not on her identity.

First, we provide an easily verifiable condition under which there exists an agent who contributes strictly positive amount with probability one in the simultaneous move game. The condition, however, requires that the distribution of types assigns strictly positive mass to the lowest type of each player.

Lemma 6. *Let* $\mu_{-k} = \prod_{j \neq k} \Pr\{\tau_j = \underline{\tau}_j\}$*, and* $i \in \operatorname{argmax}_{k \in \{1,\ldots,n\}} \underline{\tau}_k \mu_{-k}$*. Suppose that*

$$\begin{aligned} \min_{j=1,\ldots,N} \Pr\{\tau_j = \underline{\tau}_j\} &> 0, \\ \textit{and} \qquad V'(0) &> \frac{1}{\underline{\tau}_i \mu_{-i}}. \end{aligned} \tag{8}$$

Then, in any equilibrium of the simultaneous move game, at least one player contributes strictly positive amount with probability one.

Proof. Suppose there is an equilibrium of the simultaneous move game where every player contributes zero with strictly positive probability. Consider player i of type $\underline{\tau}_i$ who contributes zero. Then, the probability that the total contribution of all other players is zero with probability μ_{-i} and is bounded above by $(N-1)x(\bar{\tau})$ with probability $1 - \mu_{-i}$, where $\bar{\tau} = \max_{j=1,\ldots,N} \bar{\tau}_j$. The marginal expected payoff to player i of type $\underline{\tau}_i$ (at zero contribution) is

$$\geq \underline{\tau}_i[\mu_{-i}V'(0) + (1-\mu_{-i})V'((N-1)x(\bar{\tau}))] - 1.$$

So, if

$$\underline{\tau}[\mu_{-i}V'(0) + (1-\mu_{-i})V'((N-1)x(\bar{\tau}))] > 1, \tag{9}$$

which is implied by (8), then we immediately obtain a contradiction. **Q.E.D.**

It should be clear that condition (8) is significantly more stringent than what is required and that interior contribution by some player does not really require strictly positive mass point at the lowest type (though an easily verifiable sufficient condition may be more difficult to specify for a continuous distribution function). For specific preferences, one can provide much weaker conditions under which all players can make strictly positive contribution with probability one in a simultaneous move game. The following example considers a symmetric two-players game with quadratic utility for the public good.

Example 1. Let $N = 2$ and

$$V(G) = \begin{cases} [1-(1-G)^2], & 0 \leq G \leq 1 \\ 1, & G > 1. \end{cases}$$

Let $F_1 = F_2 = F$ where the support of F is the interval $[\underline{\tau}, \overline{\tau}]$. We do not require any probability mass point at $\underline{\tau}$. We assume:

$$\frac{1}{2} < \underline{\tau} < \overline{\tau} < 1$$

and,

$$\underline{\tau} > \frac{2}{2+m} \ , \text{ where } m = E(\frac{1}{\tau}).$$

Check that $x(\tau) = 1 - 1/(2\tau) \in (0,1), \ \tau \in [\underline{\tau}, \overline{\tau}]$. It is easy to check that assumptions 1-4 are satisfied.[9] Consider the simultaneous move contribution game. The following is a symmetric Bayesian-Nash equilibrium: each player of type $\tau \in [\underline{\tau}, \overline{\tau}]$ contributes:

$$y(\tau) = \frac{1}{2}(1 - \frac{1}{\tau} + \frac{m}{2}).$$

Observe that since $\underline{\tau} > \frac{2}{2+m}$,

$$y(\tau) > 0, \forall \tau \in [\underline{\tau}, \overline{\tau}].$$

To see that this is an equilibrium, suppose agent 2 contributes according to strategy $y(\tau)$. Then, consider agent 1's first-order condition for an interior solution to her maximization problem when she is of type τ:

$$2\tau E_{\tau_2}[(1 - y_1 - y(\tau_2))] = 1$$

i.e.,

$$\begin{aligned} \frac{1}{2\tau} &= (1-y_1) - E_{\tau_2}[y(\tau_2))] \\ &= (1-y_1) - \frac{1}{2}(1 - \frac{m}{2}), \end{aligned}$$

[9] Also note that $V'(.)$ is linear (hence, concave) and strictly decreasing ($V(.)$ is strictly concave) on $[0,1]$. Here, set $\overline{G} = 2 - \frac{1}{\tau}$.

which yields

$$y_1 = \frac{1}{2}(1 - \frac{1}{\tau} + \frac{m}{2}) = y(\tau).$$

Thus, both players playing according to the strategy $y(\tau)$ is a Bayesian-Nash equilibrium. ■

Next, we outline a condition under which the total contribution generated in the first $(N-1)$ stages of the sequential move game is strictly positive with strictly positive probability so that the antecedent of Lemma 5 holds and that one of the conditions in Proposition 2(b) and Corollary 3 hold.

To begin, let

$$\begin{aligned} \underline{\tau} &= \min\{\underline{\tau}_i : i = 1, ..., N\} \\ \bar{\tau} &= \max\{\overline{\tau}_i : i = 1, ..., N\}. \end{aligned}$$

Define the function $x(\tau)$ on the entire interval $[\underline{\tau}, \bar{\tau}]$ by:

$$x(\tau) = \text{argmax}_x \; \tau V(x) - x, \quad \tau \in [\underline{\tau}, \bar{\tau}].$$

Under assumption 1,

$$1 < \underline{\tau}_i V_i'(0) = \underline{\tau}_i V'(0), i = 1, ..., N,$$

so that $x(\tau) \in (0, \overline{G}], \forall \tau \in [\underline{\tau}, \bar{\tau}]$, and

$$\tau V'(x(\tau)) = 1.$$

Note that $x(\tau)$ is the optimal "standalone" contribution of any player of type τ i.e., $x_i(\tau_i) = x(\tau_i), \forall \tau_i \in A_i, \forall i = 1, ..., N$,where $x_i(\tau_i)$ is as defined in Section 2. Also, note that $x(\tau)$ is continuous and strictly increasing on $[\underline{\tau}, \bar{\tau}]$.

Lemma 7. *Consider any sequential contribution game $\Gamma(p)$ where $p = (p_1, ..., p_N)$ and $p_N = i$. Let $j \in \{p_1, ..., p_{N-1}\}$ be an agent such that*

$$\overline{\tau}_j \geq \overline{\tau}_{p_\ell}, \; \ell = 1, ..., N-1. \tag{10}$$

If

$$\frac{1}{\widetilde{\tau}} \int_{\underline{\tau}_i}^{\widetilde{\tau}} (1 - \frac{x(\tau)}{x(\widetilde{\tau})}) \, \mathrm{d}F_i(\tau) > \frac{1}{\overline{\tau}_j} \tag{11}$$

for some $\widetilde{\tau} \in (\underline{\tau}_i, \overline{\tau}_i)$, then in any perfect Bayesian equilibrium of $\Gamma(p)$, some agent who moves in the first $(N-1)$ stages contributes strictly positive amount with strictly positive probability i.e., free-riding is necessarily imperfect.

Proof. Suppose, to the contrary, that there exists a perfect Bayesian equilibrium of Γ where the total contribution generated in the first $(N-1)$ stages is zero almost surely. It is easy to check that in that case, the contribution generated at the end of the game is exactly the standalone contribution $x(\tau)$ of the last mover i, depending on her realized type $\tau \in A_i$.

Consider agent j (as defined in (10)) who moves in one of the first $N-1$ stages. We will show that it is strictly gainful for agent j of type $\overline{\tau}_j$ to deviate and contribute a strictly positive amount. The smoothness of the payoff function in τ then implies that there exists $\epsilon > 0$ such that deviation is strictly gainful for all types τ_j of player j lying in $[\overline{\tau}_j - \epsilon, \overline{\tau}_j]$. As $\overline{\tau}_j = \sup\{\tau : \tau \in A_j\}$ where A_j is the support of the distribution function F_j, it follows that

$$\Pr\{\tau_j \in [\overline{\tau}_j - \epsilon, \overline{\tau}_j]\} > 0$$

which would be a contradiction.

In the rest of this proof we will show that it is strictly gainful for agent j of type $\overline{\tau}_j$ to deviate and contribute an amount equal to $x(\widetilde{\tau}) > 0$ where $\widetilde{\tau}$ is as defined in the proposition.

Following such a deviation by agent j of type $\overline{\tau}_j$, the total contribution at the end of the game is at least as large as $x(\widetilde{\tau})$ if the last player's (player i's) type $\tau_i \leq \widetilde{\tau}$, while if $\tau_i > \widetilde{\tau}$ then the total contribution is at least as large as $x(\tau_i)$. Thus, the expected utility of the deviating agent j of type $\overline{\tau}_j$ is:

$$\geq \overline{\tau}_j[\int_{\underline{\tau}_i}^{\widetilde{\tau}} V(x(\widetilde{\tau}))\, \mathrm{d}F_i(\tau_i) + \int_{\widetilde{\tau}}^{\bar{\tau}_i} V(x(\tau_i))\, \mathrm{d}F_i(\tau_i)] - x(\widetilde{\tau}),$$

and so this deviation is gainful as long as:

$$\overline{\tau}_j \int_{\underline{\tau}_i}^{\overline{\tau}_i} V(x(\tau_i))\, \mathrm{d}F_i(\tau_i) < \overline{\tau}_j[\int_{\underline{\tau}_i}^{\widetilde{\tau}} V(x(\widetilde{\tau}))\, \mathrm{d}F_i(\tau_i) + \int_{\widetilde{\tau}}^{\overline{\tau}_i} V(x(\tau_i))\, \mathrm{d}F_i(\tau_i)] - x(\widetilde{\tau}),$$

which is equivalent to:

$$\overline{\tau}_j[\int_{\underline{\tau}_i}^{\widetilde{\tau}} \{V(x(\widetilde{\tau})) - V(x(\tau_i))\}\, \mathrm{d}F_i(\tau_i)] > x(\widetilde{\tau}). \tag{12}$$

We want to derive a sufficient condition such that (12) holds for some $\widetilde{\tau} \in (\underline{\tau}_i, \overline{\tau}_i)$. As V is concave,

$$\begin{aligned} V(x(\widetilde{\tau})) - V(x(\tau_i)) &\geq V'(x(\widetilde{\tau}))(x(\widetilde{\tau}) - x(\tau_i)) \\ &= \frac{1}{\widetilde{\tau}}\widetilde{\tau}V'(x(\widetilde{\tau}))(x(\widetilde{\tau}) - x(\tau_i)) \\ &= \frac{1}{\widetilde{\tau}}(x(\widetilde{\tau}) - x(\tau_i)). \end{aligned}$$

Therefore, (12) holds as long as

$$\frac{\overline{\tau}_j}{\widetilde{\tau}}[\int_{\underline{\tau}_i}^{\widetilde{\tau}} (x(\widetilde{\tau}) - x(\tau_i))\, \mathrm{d}F_i(\tau_i)] \quad > \quad x(\widetilde{\tau})$$

$$\text{i.e.,} \qquad \frac{1}{\widetilde{\tau}} \int_{\underline{\tau}_i}^{\widetilde{\tau}} (1 - \frac{x(\tau_i)}{x(\widetilde{\tau})})\, \mathrm{d}F_i(\tau_i) \quad > \quad \frac{1}{\overline{\tau}_j}$$

for some $\widetilde{\tau} \in (\underline{\tau}_i, \overline{\tau}_i)$, which follows from (11) **Q.E.D.**

Since $\int_{\underline{\tau}_i}^{\widetilde{\tau}}(1 - \frac{x(\tau_i)}{x(\widetilde{\tau})})\, \mathrm{d}F_i(\tau_i) > 0$, condition (11) is likely to be satisfied for some $\widetilde{\tau} \in (\underline{\tau}_i, \overline{\tau}_i)$ if $\overline{\tau}_j$ is large enough. Further, larger the probability mass of the distribution of types in a neighborhood of $\underline{\tau}_i$, the easier it is for this condition to hold.

The economic intuition is that greater the likelihood that the last mover has very low valuation for the public good, greater the risk faced by an early mover with high valuation for the public good when she tries to fully free ride on the last mover because that may, in certain states of nature, lead to very low provision of the public good. This creates incentive for an early mover with high valuation to make a strictly positive contribution. It is easy to see that (11) is more likely to hold if an early mover has significantly "higher" distribution of valuations (types) than the last mover. Indeed, even under complete information, an early mover may not free ride on the last mover and may contribute a strictly positive amount if her valuation of the public good is significantly higher than the last mover.

If agents are identical, then under complete information, early movers always free ride fully on the last mover. With incomplete information however, even if all agents have identical distribution of types, early movers may not fully free ride on the last mover – this, in fact, is a crucial difference between the complete and incomplete information sequential games. We illustrate this in the next example where all agents have identical distribution of types and for a specific class of distribution functions, we show that (11) can be satisfied (so that some early mover contributes a strictly positive amount with strictly positive probability).

Example 2. Suppose that $F_i = F, \forall i \in \{1, ..., N\}$, where the support of F is the interval $[\underline{\tau}, \overline{\tau}]$ and F has the following structure:

$$F(\tau) = \begin{cases} (1-\epsilon)G(\tau), & \underline{\tau} \leq \tau \leq \widetilde{\tau} \\ (1-\epsilon) + \epsilon H(\tau), & \widetilde{\tau} \leq \tau \leq \overline{\tau}, \end{cases}$$

where $0 < \epsilon < 1$, $\widetilde{\tau} \in (\underline{\tau}, \overline{\tau})$, $G(\tau)$ is any probability distribution function with support $[\underline{\tau}, \widetilde{\tau}]$ satisfying

$$0 \leq G(\underline{\tau}) < G(\widetilde{\tau}) = 1,$$

and $H(\tau)$ is a probability distribution function with support $[\widetilde{\tau}, \overline{\tau}]$, satisfying $H(\widetilde{\tau}) = 0$.

Then, (11) reduces to the requirement that:

$$\begin{aligned} 1 &< \frac{\overline{\tau}}{\widetilde{\tau}}[\int_{\underline{\tau}}^{\widetilde{\tau}} (1 - \frac{x(\tau)}{x(\widetilde{\tau})}) \, \mathrm{d}F(\tau)] \\ &= \overline{\tau}\left\{\frac{1-\epsilon}{\widetilde{\tau}}[\int_{\underline{\tau}}^{\widetilde{\tau}} (1 - \frac{x(\tau)}{x(\widetilde{\tau})}) \, \mathrm{d}G(\tau)]\right\} \\ &= \overline{\tau}\left\{\frac{1-\epsilon}{\widetilde{\tau}}\left[1 - \frac{1}{x(\widetilde{\tau})}\int_{\underline{\tau}}^{\widetilde{\tau}} x(\tau) \, \mathrm{d}G(\tau)\right]\right\}. \end{aligned} \tag{13}$$

As $\int_{\underline{\tau}}^{\widetilde{\tau}} x(\tau) \, \mathrm{d}G(\tau) < x(\widetilde{\tau})$,

$$[1 - \frac{1}{x(\widetilde{\tau})}\int_{\underline{\tau}}^{\widetilde{\tau}} x(\tau) \, \mathrm{d}G(\tau)] > 0.$$

Since $H, \overline{\tau}$ are independent of the distribution function G whose support is $[\underline{\tau}, \widetilde{\tau}]$,

$$\left\{\frac{1-\epsilon}{\widetilde{\tau}}\left[1 - \frac{1}{x(\widetilde{\tau})}\int_{\underline{\tau}}^{\widetilde{\tau}} x(\tau) \, \mathrm{d}G(\tau)\right]\right\}$$

is independent of H and $\overline{\tau}$ in particular, so (13) holds if $\overline{\tau}$ is large enough. ■

Finally, we note that condition (13) that ensures strictly positive contribution by some early mover in the sequential game is consistent with the last mover of the sequential game making strictly positive contribution with probability one in the simultaneous move game (as required in some of the results of the previous section). To illustrate this, consider Example 2 above. The sufficient condition (8) in Lemma 6 that ensures that some player contributes strictly positive amount almost surely is satisfied as long as:

$$\eta = \lim_{\tau \downarrow \underline{\tau}} G(\tau) > 0$$

and

$$V'(0) > \frac{1}{\underline{\tau}((1-\epsilon)\eta)^{N-1}}$$

and these are perfectly consistent with $\overline{\tau}$ being large enough so that (13) holds.

7 A special case: two agents, two types

In this section, we discuss a special case of our model with two agents ($N = 2$) and two potential types. We use this to make two important points.

First, though our general results in Section 5 require that at least some agent contributes strictly positive amount with probability one in the simultaneous move game (in order to obtain weakly or strictly higher expected total contribution in the sequential move game), this is by no means necessary. We show that even if all agents contribute zero with strictly positive probability in the simultaneous move game, the sequential move game may still generate *strictly* higher expected total contribution.

Second, our general results in Section 5 show that when *the last mover* is the one who contributes strictly positive amount almost surely in the simultaneous move game, the sequential game generates weakly higher expected total contribution (and under additional conditions, strictly higher contributions). We show that even if the agent who contributes strictly positive amount with probability one in the simultaneous move game moves early in the sequential game and the last mover is an agent who contributes zero with strictly positive probability in the simultaneous move game, the sequential game may still generate strictly higher expected total contribution.

In this section, we assume that Assumptions 1 – 4 hold so that

$$V_i(G) = V(G), \quad i = 1, 2,$$

and further restrict:

$$A_i = \{\tau_L, \tau_H\}, \quad \text{with } 0 < \pi_i = \Pr[\tau_i = \tau_H] < 1, \quad i = 1, 2.$$

Proposition 1 shows that there is a sequential game that generates weakly higher expected total contribution than the simultaneous move game as long as there is at least one agent who contributes strictly positive amount in the simultaneous move game for (almost) every realization of her type. If, in addition, the requirements of Proposition 2 are satisfied, then the sequential game generates strictly higher expected total contributions. If the simultaneous move game is such that every agent contributes zero with positive probability, then the results outlined in the previous section no longer apply. However, for the special case considered in this section, we can show that under certain conditions, the sequential contribution game generates higher expected total contributions even though the equilibrium in the simultaneous move game is one where both agents contribute zero when their realized type is τ_L. Thus, the interiority of equilibrium contributions (for some agent) is not necessary for higher contributions under the sequential form of the contribution game.

Recall that $x(\tau)$ is the "standalone" contribution of type τ as defined earlier.

Proposition 3. *Suppose that*

$$\frac{x(\tau_H)}{x(\tau_L)} < \min\{\frac{1}{\pi_i}, 2\pi_k + \frac{1-\pi_k}{\pi_i}\}, \tag{14}$$

where

$$\pi_k = \max\{\pi_1, \pi_2\}, \quad \textit{and } i \neq k.$$

Then, the expected total contribution generated in any perfect Bayesian equilibrium of the sequential game $\Gamma(p)$, where $p = (p_1, p_2)$ and $p_2 = k$, is strictly greater than that generated in any Bayesian-Nash equilibrium of the simultaneous contribution game where both agents contribute zero when their realized type is τ_L.

The proof appears in the Appendix.

For $\pi_1 = \pi_2 = \frac{1}{2}$, (14) reduces to the requirement that $x(\tau_H) < 2x(\tau_L)$, which is quite easily satisfied as long as τ_L and τ_H are not too far apart.

Next, we point out that even if the sequential move game is such that the agent (or agents) who contribute strictly positive amount for all realization of types in the simultaneous move game move earlier than other agents, the sequential move game may still generate higher expected total contribution than the simultaneous move game. We illustrate this in the example (summarized) below where within the framework of two agents and two types considered in this section, we choose a specific quadratic functional form for the $V(.)$ function.

Example 3. Let

$$V(G) = \begin{cases} [1-(1-G)^2], & 0 \leq G \leq 1 \\ 1, & G > 1. \end{cases}$$

Here, $x(\tau) = 1 - 1/(2\tau)$, $\tau = \tau_H, \tau_L$. Note that $V'(.)$ is linear (hence, concave) and strictly decreasing ($V(.)$ is strictly concave) on $[0, 1]$. Here, $\overline{G} = 2 - \frac{1}{\tau_H}$. Assumptions 1-4 are satisfied as long as $\tau_L > \frac{1}{2}$, $\tau_H < 1$.

Suppose $\pi_j > \pi_i$. Consider the sequential move game $\Gamma(p)$ where $p = (j, i)$ i.e., player j moves first and player i moves next. It can be checked that if

$$\pi_i \leq \max\left\{0, 1 - \frac{\tau_L}{\tau_H}\left[2\tau_L + \sqrt{4\tau_L^2 - 1}\right]\right\}, \qquad (15)$$

then the first-mover (player j) of τ_H-type will make a strictly positive contribution in the (unique perfect Bayesian equilibrium) sequential move game and the expected total contribution generated is:

$$\tilde{z} = (\pi_i + \pi_j)x(\tau_H) + (1 - \pi_i)(1 - \pi_j)x(\tau_L) - \pi_i\pi_j.$$

In the simultaneous move game, it can be shown that there is a unique Bayesian-Nash equilibrium and in this equilibrium, agent j makes a strictly positive contribution for both realization of types, while agent i makes a strictly positive contribution only if it is of τ_H-type. The expected total contribution in the simultaneous move game $\tilde{y}$ is exactly equal to the expected standalone contribution of agent j i.e., $\tilde{y} = \pi_j x(\tau_H) + (1 - \pi_j)x(\tau_L)$. Observe that $\tilde{z} - \tilde{y} = [\frac{1-\pi_j}{\tau_L} - \frac{1}{\tau_H}][\frac{\pi_i}{2}] > 0$ if

$$\pi_j < 1 - \frac{\tau_L}{\tau_H}. \qquad (16)$$

Thus, even though the agent who has strictly higher probability of being τ_H-type and contributes strictly positive amount almost surely in the simultaneous move game moves first in the sequential move game and the agent who does not contribute strictly positive amount almost surely in the simultaneous move game is the last mover in the sequential move game, the sequential game generates (strictly) higher expected total contribution relative to the simultaneous move game if (15) and (16) hold. ■

Appendix

Proof of Lemma 2. First, observe that in any equilibrium of the simultaneous move game, the first order condition for any player j of type τ is given by:

$$\begin{aligned} E[\tau_j V_j'(y_j(\tau_j) + \sum_{k \neq j} y_k)] &= 1, \text{ if } y_j(\tau_j) > 0 \\ &\leq 1, \text{ if } y_j(\tau_j) = 0 \end{aligned}$$

(where the expectation is taken with respect to the distribution of $\sum_{k \neq j} y_k$). Using the strict concavity of $V_j(.)$ in the relevant range, it is easy to check that each player j's equilibrium contribution is weakly increasing in his type τ_j and, further, strictly increasing over the range of types τ_j where $y_j(\tau_j) > 0$. Therefore, for any fixed player i, the lower bound (infimum) of the support of the distribution of $y_{-i} = \sum_{j \neq i} y_j$, the total equilibrium contributions of all players other than i (whether or not they contribute a strictly positive amount), is given by their total contribution in the state where $\tau_j = \underline{\tau}, \forall j \neq i$, and the upper bound (supremum) of the support of the distribution of y_{-i} is given by their total contribution in the state where $\tau_j = \bar{\tau}, \forall j \neq i$. These two bounds cannot be identical unless all players other than player i contribute zero with probability one, and the latter is not possible given that $\Pr\{\tau_j : y_j(\tau_j) > 0\}$ for some $j \neq i$. **Q.E.D.**

Proof of Proposition 3. Let $y_1(\tau), y_2(\tau), \tau = \tau_H, \tau_L$, denote the equilibrium contributions of the two agents in the simultaneous move game where

$$y_1(\tau_L) = y_2(\tau_L) = 0.$$

The first-order condition of maximization for agent i of type τ_L yields for $i, j = 1, 2,\ j \neq i$,

$$\pi_j \tau_L V'(y_j(\tau_H)) + (1 - \pi_j)\tau_L V'(0) \leq 1. \tag{17}$$

Using Assumption 2,

$$\pi_j \tau_L V'(x(\tau_L)) + (1 - \pi_j)\tau_L V'(0) > 1, \tag{18}$$

which implies that (comparing (17) and (18))

$$y_j(\tau_H) > x(\tau_L) > 0,\ j = 1, 2.$$

From the first-order condition for agent i of type τ_H we have for $i, j = 1, 2,\ j \neq i$,

$$\pi_j \tau_H V'(y_i(\tau_H) + y_j(\tau_H)) + (1 - \pi_j)\tau_H V'(y_i(\tau_H)) = 1 \tag{19}$$

so that

$$\tau_H V'(y_i(\tau_H)) > 1,\ \ i = 1, 2 \tag{20}$$

and therefore $y_i(\tau_H) < x(\tau_H)$, $i = 1, 2$. Thus,

$$x(\tau_L) < y_j(\tau_H) < x(\tau_H), \quad j = 1, 2. \tag{21}$$

The expected total contribution generated in this game is

$$[\pi_1 y_1(\tau_H) + \pi_2 y_2(\tau_H)].$$

Further, from the first-order conditions:

$$\pi_1 V'(y_1(\tau_H) + y_2(\tau_H)) + (1 - \pi_1)V'(y_2(\tau_H)) = \frac{1}{\tau_H} = V'(x(\tau_H)) \tag{22}$$

$$\pi_2 V'(y_1(\tau_H) + y_2(\tau_H)) + (1 - \pi_2)V'(y_1(\tau_H)) = \frac{1}{\tau_H} = V'(x(\tau_H)) \tag{23}$$

and using the concavity of $V'(.)$ on $[0, \overline{G}]$ and Jensen's inequality we have:

$$\pi_1\, y_1(\tau_H) + y_2(\tau_H) \leq x(\tau_H) \tag{24}$$

$$\pi_2\, y_2(\tau_H) + y_1(\tau_H) \leq x(\tau_H). \tag{25}$$

Also, from (22) and (23), $V'(y_1(\tau_H) + y_2(\tau_H)) < V'(x(\tau_H))$ so that

$$y_1(\tau_H) + y_2(\tau_H) > x(\tau_H). \tag{26}$$

From (21) and (26),

$$y_1(\tau_H) + y_2(\tau_H) > \max\{x(\tau_H), 2x(\tau_L)\}. \tag{27}$$

Multiply (24) by π_2 and (25) by π_1 and add to obtain

$$\begin{aligned} & \pi_1\pi_2\, y_1(\tau_H) + \pi_2\, y_2(\tau_H) + \pi_1\pi_2\, y_2(\tau_H) + \pi_1\, y_1(\tau_H) \\ \leq \quad & (\pi_1 + \pi_2)x(\tau_H), \end{aligned}$$

implying

$$\begin{aligned} & \pi_1\, y_1(\tau_H) + \pi_2\, y_2(\tau_H) \\ \leq \quad & (\pi_1 + \pi_2)x(\tau_H) - \pi_1\pi_2[y_1(\tau_H) + y_2(\tau_H)] \\ = \quad & [\pi_k x(\tau_H) + (1 - \pi_k)x(\tau_L)] + [\pi_i\{x(\tau_H) - \pi_k[y_i(\tau_H) + y_k(\tau_H)]\}] - (1 - \pi_k)x(\tau_L),\ i \neq k \\ \underbrace{<}_{\text{using (27)}} \quad & \pi_k x(\tau_H) + (1 - \pi_k)x(\tau_L) + [\pi_i\{x(\tau_H) - \pi_k \cdot \max\{x(\tau_H), 2x(\tau_L)\}\}] - (1 - \pi_k)x(\tau_L) \\ < \quad & \pi_k x(\tau_H) + (1 - \pi_k)x(\tau_L). \qquad \text{(using (14))} \end{aligned}$$

Therefore, the expected total contribution in this equilibrium of the simultaneous move game $\pi_1\, y_1(\tau_H) + \pi_2\, y_2(\tau_H) < \pi_k x(\tau_H) + (1 - \pi_k)x(\tau_L)$, the expected standalone contribution of agent k. Finally, from Lemma 4, we know that in the sequential move game $\Gamma(p)$ where $p = (p_1, p_2), p_2 = k$ i.e., agent k is the last mover, the expected total contribution generated is at least as large as the expected standalone contribution $[\pi_k x(\tau_H) + (1 - \pi_k)x(\tau_L)]$ of agent k. The proposition follows. **Q.E.D.**

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