Priority-augmented House Allocation*

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Abstract

We consider the standard indivisible object allocation problem without monetary transfer and allow each object to have a weak priority over agents. It is well known that generally in such a problem stability (or no justified-envy) is not compatible with efficiency. We characterize the priority structures for which a stable and efficient assignment always exists, as well as the priority structures which admit a stable, efficient and (group) strategy-proof rule. While house allocation and housing market are two classical allocation problems that admit a stable, efficient and group strategy-proof rule, any priority-augmented allocation problem with more than three objects admits such a rule if and only if it is decomposable into a sequence of subproblems, each of which has the house allocation or the housing market structure.

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1 Introduction

In a priority-augmented allocation problem, each agent has a preference ordering over a set of heterogeneous and indivisible objects and each object also comes with a priority ordering over agents. Each agent will be assigned at most one object without monetary transfer. Common real-world applications include assigning students to public schools, faculty members to offices, on-campus housing allocation and so on. We say an assignment respects the priorities or there is *no justified-envy* if there is no such situation that one agent envies another's assignment for which the first agent has a strictly higher priority. Together with individual rationality and nonwastefulness, this fairness notion is equivalent to the stability concept in the corresponding two-sided matching problem if priorities are interpreted as preferences of objects. As shown in Roth (1982), the man-optimal stable matching in the marriage problem is only weakly Pareto optimal but not strongly Pareto optimal for all the men, which implies in general stability and efficiency are not compatible in priority-augmented allocation. Therefore, we are interested in characterizing the class of solvable problems for which a stable and efficient rule exists.[1]

The answer is already known when priority orderings are strict. Gale and Shapley (1962)'s deferred acceptance (DA) algorithm is the "best rule": it yields the unique stable matching that Pareto dominates any other stable matching, and it is also strategy-proof (Dubins and Freedman, 1981, Roth, 1982). Ergin (2002) shows that DA is efficient (group strategy-proof) if and only if the priority structure is acyclic. Thus acyclicity characterizes the priority structures under which an efficient and stable assignment exists for any preference profile, as well as the priority structures that admit a stable, efficient and group strategy-proof rule.[2]

However, indifferences in priority rankings are common in real world applications: in a school choice problem, a student's priority at a particular school could be only determined by the district and sibling rule; In on-campus dormitory allocation, current residents or senior students are usually given higher priorities than others. When ties in priorities are allowed, acyclicity no longer guarantees the existence of a stable and efficient assignment. In this study

[1] We focus on characterizing priority structures in this study. Heo (2014) gives the maximal preference domain in which stability and efficiency are compatible.

[2] There are several related characterizations of the strict priority structures. Kesten (2006) defines a stronger acyclicity condition and shows that DA is equivalent to Gale's top trading cycle if and only if the priority structure is acyclic. Both Ergin (2002) and Kesten (2006) consider the many-to-one case (there are many copies of each object) and in the many-to-many setting (each agent also has multi-unit demand), Kojima (2013) shows that stability is compatible with efficiency or strategy-proofness if and only if the priority structure is essentially homogeneous.

we consider the allocation problem with weak priorities in the one-to-one matching context, we show that our *non-reversal* condition is both necessary and sufficient for the existence of a stable and efficient rule, but the set of priority structures that admit a stable, efficient and strategy-proof rule is strictly smaller.[3] Requiring group strategy-proofness further reduces the "maximal domain" of priority structures, which can be characterized by *strong non-reversal*. For the sufficiency parts of the characterizations we introduce the *priority set rules*. When a group of agents are ranked as high as all the other agents for any object, we call them a priority set and can conduct allocation to this priority set first without violating stability constraints. It turns out that non-reversal imposes strong structural requirements on a subproblem induced by the smallest priority set, which can only take one of the three forms: house allocation, housing market, and indifference at the top (IT) which is a dual structure to the housing market. In order to select a stable and efficient assignment, serial dictatorship and top trading cycle can be used for the first two structures, and a modified serial dictatorship algorithm is designed for the IT structure. After agents in the smallest priority set leave the problem with their assignments, we find the smallest priority set of the reduced problem and repeat this process iteratively.[4]

The priority set rules can elicit true preferences for the house allocation and housing market structures, but for an IT structure with more than three agents, stable and efficient assignments cannot be selected through a strategy-proof rule. Moreover, when group strategy-proofness is imposed, the IT structures are eliminated from any solvable problem if there are at least four objects. Therefore, the two baseline allocation problems, house allocation (Hylland and Zeckhauser, 1979) and housing market (Shapley and Scarf, 1974), are not merely special structures that admit an efficient, stable and group strategy-proof rule. In general any priority-augmented allocation problem admits such a rule if and only if it can be decomposed as a sequence of subproblems with the house allocation or housing market structure during the iteration process of the priority set rules.

Priority-augmented allocation has been studied extensively in the context of the school choice problem, starting from Abdulkadiroğlu and Sönmez (2003). Given coarse priority rank-

[3]We maintain the usual assumption of strict preferences. For the allocation problem on the full preference domain, see Bogomolnaia et al. (2005) for the house allocation problem, Ehlers (2014) for the housing market problem.

[4]The idea of priority set rules is similar to the method employed in Ehlers and Westkamp (2011), where for the same allocation problem they consider the existence of a strategy-proof constrained efficient rule. A rule is constrained efficient if it yields a stable assignment that cannot be Pareto dominated by any other stable assignment.

ings, a common practice is to break the ties in priorities randomly such that DA can be applied, but such a rule is not even constrained efficient for strong non-reversal priority structures (Erdil and Ergin, 2008).[5] [6] While DA is the only constrained efficient rule under strict priorities, constrained efficient assignment is in general not unique when ties are allowed. Ehlers and Erdil (2010) shows that the constrained efficient correspondence is efficient if and only if the priority structure is strongly acyclic, but this requirement is more stringent than strong non-reversal.[7] However, despite of the deficiencies of both DA (with fixed tiebreaking) and the constrained efficient rule under weak priorities, a top trading cycle approach has more satisfying performance. We show that certain selections from the class of hierarchical exchange rules by Pápai (2000), which are the only efficient, group strategy-proof and reallocation-proof rules in the standard one-to-one assignment problem, can achieve stability as long as there exists a stable, efficient and group strategy-proof rule.[8] Hierarchical exchange rules generalize Gale's top trading cycle algorithm and the priority set rules are equivalent to a subclass of hierarchical exchange rules in the strong non-reversal priority domain.

In the next section we set up the model and define useful concepts. Section 3 deals with the existence of a stable and efficient rule and Section 4 presents the results when strategy-proofness and group strategy-proofness are imposed. Section 5 provides further discussion of the results and then Section 6 concludes. All the proofs are contained in Appendix A.

2 Preliminaries

Let N be a finite set of agents and H a finite set of objects (houses). Each agent $i \in N$ has a complete, transitive and antisymmetric **preference relation** R_i over $H \cup \{i\}$ with P_i denoting its asymmetric component, a house $a \in H$ is **acceptable** to i if $a P_i i$. Then $R = (R_i)_{i \in N}$ denotes a **preference profile** for all the agents. Each house $a \in H$ has a complete and transitive **prior-**

[5] DA with fixed tiebreaking rule is currently used in many school choice programs in the U.S.. Economists Abdulkadiroğlu, Pathak, Roth and Sönmez assisted in redesigning the school choice program and adopting the DA algorithm with fixed tiebreaking rule in New York City (in 2003) and Boston (in 2006). See Abdulkadiroğlu et al. (2005a) and Abdulkadiroğlu et al. (2005b).

[6] Erdil and Ergin (2008) gives an example of housing market structure and shows that DA with fixed tiebreaking is not constrained efficient (indeed, a stable, efficient and group strategy-proof rule exists for such a problem (Gale's top trading cycle algorithm) but DA with any fixed tiebreaking is not efficient). They introduce the stable improvement cycles algorithm, which can always select a constrained efficient assignment in polynomial-time.

[7] So for some priority structure that fails to satisfy strong acyclicity, not every constrained efficient assignment is efficient, but there could exist some efficient selection from the set of constrained efficient assignments.

[8] Reallocation-proofness rules out the possibility that two individuals can gain by jointly manipulating the outcome and swapping objects ex post.

ity ordering $\succeq_a$ over N, with $\succ_a$ and $\sim_a$ denoting its asymmetric and symmetric component respectively.[9] A **priority structure** $\succeq = (\succeq_a)_{a \in H}$ is a profile of priority orderings. An **assignment** or **matching** is a one-one function $\mu : N \to H \cup N$ such that for $\forall i \in N, \mu(i) \in H \cup \{i\}$. Given the problem $\{N, H, \succeq\}$, a **rule** or **mechanism** is a function f that associates an assignment $f(R)$ to each preference profile R.

An assignment ν **Pareto dominates** μ if $\nu(i) R_i \mu(i)$ for all i and $\nu(j) P_j \mu(j)$ for some $j \in N$, and an assignment μ is **efficient** if it cannot be Pareto dominated by any other assignment. μ is **stable** if it satisfies the three following conditions: *(i)* **respecting priorities (no justified-envy)**, $\mu(j) P_i \mu(i)$ implies $j \succeq_{\mu(j)} i$ for all i, j; *(ii)* **individually rational**, $\mu(i) R_i i$ for all i; *(iii)* **nonwasteful**, for any $h \in H$, $\mu^{-1}(h) = \phi$ implies $\mu(i) R_i h$ for all i.[10]

A rule f is efficient (resp., stable) if for any preference profile R, $f(R)$ is efficient (resp., stable). f is **strategy-proof** if it is a weakly dominant strategy for each agent to report true preference in the associated preference revelation game, i.e., given any R, i and R_i', $f_i(R) R_i f_i(R_i', R_{-i})$. f is **nonbossy** if no agent can change others' assignments without affecting her own assignment: given any R, i and R_i', $f_i(R) = f_i(R_i', R_{-i})$ implies $f(R) = f(R_i', R_{-i})$. f is **group strategy-proof** if no group of agents can jointly manipulate: given any R, there are no $N' \subseteq N$, $R_{N'}'$ such that $f_i(R_{N'}', R_{-N'}) R_i f_i(R)$ for all $i \in N'$, and $f_j(R_{N'}', R_{-N'}) P_j f_j(R)$ for some $j \in N'$. We also have this weaker form of group strategy-proofness: f is **weakly group strategy-proof** if given any R, there are no $N' \subseteq N$, $R_{N'}'$ such that $f_i(R_{N'}', R_{-N'}) P_i f_i(R)$ for all $i \in N'$.

Lemma 1 (Pápai, 2000) *f is group strategy-proof if and only if f is strategy-proof and nonbossy.*

When $\succeq_a$ is antisymmetric for any $a \in H$, i.e., priorities are strict, the (agent-proposing) deferred acceptance (DA) algorithm of Gale and Shapley (1962) yields the unique stable assignment that Pareto dominates any other stable assignment. DA is stable and strategy-proof, but may not be efficient in the context of priority-augmented allocation. Ergin (2002) characterizes the priority structures in which DA is efficient by an acyclicity condition.

Definition 1 A **cycle** consists of three agents i, j, k and two houses a, b such that $i \succ_a j \succ_a k \succ_b i$, and $\succeq$ is **Ergin-acyclic** if there does not exist any cycle.[11]

[9] We abuse the notation a little bit when there is no confusion: given $A \subseteq N, B \subseteq N, \widetilde{H} \subseteq H$, denote $A \succ_{\widetilde{H}} B$ if $i \succ_a j$ for $\forall i \in A, j \in B, a \in \widetilde{H}$, and similarly we can define $i \succ_a A, A \succeq_{\widetilde{H}} B, i \succeq_{\widetilde{H}} B$ and so on.

[10] The stability concept here is equivalent to the one in the corresponding two-sided matching problem where priorities are interpreted as preferences of houses. Here it is assumed each house finds all the agents "acceptable", and respecting priorities and nonwastefulness are equivalent to "no blocking pairs".

[11] The scarcity condition in the original definition is omitted here since it is always satisfied in our one-to-one

Theorem 1 *(Ergin, 2002) Given the problem $\{N, H, \succeq\}$, assume $\succeq_a$ is antisymmetric for any $a \in H$. Then the following are equivalent:*
(i) DA is efficient,
(ii) DA is group strategy-proof,
(iii) $\succeq$ is Ergin-acyclic.

Thus in the case of strict priorities, Ergin-acyclicity characterizes the priority structures in which a stable and efficient assignment always exists, as well as the priority structures in which a stable, efficient and group strategy-proof rule exists.

3 The existence of a stable and efficient assignment

We first consider the following motivating example from Ehlers and Erdil (2010), which illustrates the tension between efficiency and respecting (weak) priorities.

Example 1 (Ehlers and Erdil, 2010) Suppose there are two houses a, b and three agents i, j, k. The priority structure and preference profile are as following:

$\succeq_a$	$\succeq_b$
i, j	k
k	i, j

R_i	R_j	R_k
b	b	a
a	a	b
i	j	k

where for both a and b, i and j are ranked equally. It can be easily shown that in this example stability induces efficiency loss. Consider any stable assignment μ, we first observe that $\mu(k) \neq a$, otherwise at least one of i and j is self-assigned and her priority for a is violated. Secondly we must have $\mu(k) = b$ otherwise either b is wasted or k 's priority for b is violated. Then a will be assigned to either i or j otherwise it is wasted. Suppose $\mu(i) = a$. Then clearly i and k can exchange their assignments under μ and both would obtain their first choice. Hence μ is not efficient, stable and efficient assignment does not exist in this example.

The priority structure here satisfies Ergin-acyclicity so we need a stronger restriction on priorities to ensure the stability constraints will not induce welfare loss. Actually this example

setting.

is representative: efficiency and stability are compatible as long as there does not exist such form of priority relation.

Definition 2 A **priority reversal** consists of three distinct agents i, j, k and two distinct houses a, b such that $\{i, j\} \succ_a k \succ_b \{i, j\}$. A priority structure $\succeq$ satisfies **non-reversal** if there is no priority reversal.

Notice that non-reversal implies Ergin-acyclicity. Suppose there is a cycle $i \succ_a j \succ_a k \succ_b i$, then consider j 's priority for b. If $j \succeq_b k$, we have $\{j,k\} \succ_b i \succ_a \{j,k\}$. If $k \succ_b j$, we have $\{i, j\} \succ_a k \succ_b \{i, j\}$. So there exists a priority reversal.

It can be easily seen from Example 1 that non-reversal is necessary for the compatibility of efficiency and stability. For the rest of this section, we focus on constructing a rule which can always select a stable and efficient assignment when non-reversal is satisfied. The basic strategy is to decompose this allocation problem into a sequence of smaller and easier problems. When a subset of agents are ranked as high as anyone outside this subset for all the houses, we can allocate houses to this subset first without worrying about violation of priorities, and once these agents leave the problem with their assignments, we can repeat this process for the reduced problem iteratively. We define such set of agents first.

Definition 3 Given $\widetilde{N} \subseteq N, \widetilde{H} \subseteq H, \widetilde{N} \neq \phi, \widetilde{H} \neq \phi$, a nonempty set $S \subseteq \widetilde{N}$ is a **priority set** for the subproblem $\{\widetilde{N}, \widetilde{H}, \succeq_{\widetilde{H}}\}$ if $\forall i \in S, j \in \widetilde{N} \setminus S, i \succeq_a j$ for all $a \in \widetilde{H}$ and $i \succ_b j$ for some $b \in \widetilde{H}$, or denoted as $i \succ\succeq_{\widetilde{H}} j$.

Lemma 2 *There always exists a priority set, and if S_1 and S_2 are two priority sets, $S_1 \cap S_2$ is a priority set.*

The problem is finite, so Lemma 2 implies that if we take the intersection of all the priority sets then we can find a unique priority set with the smallest number of agents.

Definition 4 A is the **smallest priority set** for $\{\widetilde{N}, \widetilde{H}, \succeq_{\widetilde{H}}\}$ if A is a priority set and for any priority set S for $\{\widetilde{N}, \widetilde{H}, \succeq_{\widetilde{H}}\}$, $A \subseteq S$. A subproblem $\{\widetilde{N}, \widetilde{H}, \succeq_{\widetilde{H}}\}$ is **minimal** if $\widetilde{N}$ is the smallest priority set for $\{\widetilde{N}, \widetilde{H}, \succeq_{\widetilde{H}}\}$.

It turns out that non-reversal imposes strong restrictions on the structure of a minimal subproblem, which can only take three possible forms: each house ranks all the agents equally, every agent has some house that only ranks her at the top, or every agent has some house that only puts her in the bottom.

Lemma 3 $\succeq$ *satisfies non-reversal if and only if any minimal subproblem* $\{\widetilde{N},\widetilde{H},\succeq_{\widetilde{H}}\}$ *satisfies one of the following structures (let* $I_{\widetilde{H}}=\{a\in\widetilde{H}: i\sim_a j\ \forall i,j\in\widetilde{N}\}$*):*
(i) (House allocation) $\widetilde{H}=I_{\widetilde{H}}$,
(ii) (Housing market) For each $i\in\widetilde{N}$*, there exists nonempty* $\mathscr{U}(i)\subseteq\widetilde{H}$*, such that for any* $a\in\mathscr{U}(i)$*,* $i\succ_a j$ *for all* $j\in\widetilde{N}\setminus\{i\}$ *and* $j\sim_a k$ *for all* $j,k\in\widetilde{N}\setminus\{i\}$*. And* $\widetilde{H}=I_{\widetilde{H}}\cup\{\cup_{i\in\widetilde{N}}\mathscr{U}(i)\}$,
(iii) (Indifference at the Top, or IT) For each $i\in\widetilde{N}$*, there exists nonempty* $\mathscr{D}(i)\subseteq\widetilde{H}$ *such that for any* $a\in\mathscr{D}(i)$*,* $j\succ_a i$ *for all* $j\in\widetilde{N}\setminus\{i\}$ *and* $j\sim_a k$ *for all* $j,k\in\widetilde{N}\setminus\{i\}$*.* $\widetilde{H}=I_{\widetilde{H}}\cup\{\cup_{i\in\widetilde{N}}\mathscr{D}(i)\}$*.*[12]

A minimal subproblem with the house allocation structure is exactly a house allocation problem (Hylland and Zeckhauser, 1979). Every agent is ranked equally by all the houses and the only stability constraints in effect are individual rationality and nonwastefulness. The simple **serial dictatorship** is stable, efficient and group strategy-proof (Svensson, 1994, Svensson, 1999). Specifically, for a subproblem $\{\widetilde{N},\widetilde{H},\succeq_{\widetilde{H}}\}$ with the house allocation structure, given $R_{\widetilde{N}}$ we fix an ordering σ of agents ($\sigma:\{1,2,...,|\widetilde{N}|\}\rightarrow\widetilde{N}$) and let them choose their best available option successively according to σ, the resulting assignment is denoted as $f^{SD}(\sigma,R_{\widetilde{N}})$.

Before discussing the other two structures, we give an example of them when $|\widetilde{H}|=8$ and the smallest priority set consists of four agents.

Example 2 A housing market structure:

$\succeq_a$	$\succeq_b$	$\succeq_c$	$\succeq_d$	$\succeq_e$	$\succeq_f$	$\succeq_g$	$\succeq_h$
1	2	3	4	1234	1234	1	4
234	134	124	123			234	123

where $\mathscr{U}(1)=\{a,g\}$, $\mathscr{U}(2)=\{b\}$, $\mathscr{U}(3)=\{c\}$, $\mathscr{U}(4)=\{d,h\}$, $I_{\widetilde{H}}=\{e,f\}$
An IT structure:

$\succeq_a$	$\succeq_b$	$\succeq_c$	$\succeq_d$	$\succeq_e$	$\succeq_f$	$\succeq_g$	$\succeq_h$
1234	234	134	124	123	124	124	1234
	1	2	3	4	3	3	

[12] We say a subproblem $\{\widetilde{N},\widetilde{H},\succeq_{\widetilde{H}}\}$ with $|\widetilde{N}|=1$ has the house allocation structure. When $\succeq$ satisfies non-reversal, a minimal subproblem $\{\widetilde{N},\widetilde{H},\succeq_{\widetilde{H}}\}$ with $|\widetilde{N}|=2$ satisfies both *(ii)* and *(iii)* if it does not have the house allocation structure, in this case we say it has the housing market structure. So an IT structure must have at least three agents.

where $\mathscr{D}(1)=\{b\}, \mathscr{D}(2)=\{c\}, \mathscr{D}(3)=\{d,f,g\}, \mathscr{D}(4)=\{e\}$, $I_{\widetilde{H}}=\{a,h\}$

The housing market structure has features of both the housing market problem (Shapley and Scarf, 1974) and house allocation with existing tenants (Abdulkadiroğlu and Sönmez, 1999), since each agent i can be considered to have initial endowment $\mathscr{U}(i)$ but there could also be a set of vacant houses $I_{\widetilde{H}}$. When there are equal number of agents and houses, a housing market structure is exactly a housing market problem. The **top trading cycle mechanism (TTC)** of Abdulkadiroğlu and Sönmez (1999) can be applied to the housing market structure.[13] For a subproblem $\{\widetilde{N}, \widetilde{H}, \succeq_{\widetilde{H}}\}$ with this structure, given σ, $f^{TTC}(\sigma,\cdot)$ is defined as follows:

Step 1. Denote $t_1=\sigma(1)$. Let agent i's initial endowment be $E_i^1=\mathscr{U}(i)$ for $i\neq t_1$, and $E_{t_1}^1=\mathscr{U}(t_1)\cup I_{\widetilde{H}}$. Given $R_{\widetilde{N}}$, let agents start a top trading cycle exchange with respect to E^1: each agent points to the owner of her favorite house (or herself if all the available houses are not acceptable), and there exists at least one cycle since $\widetilde{N}$ is finite. Let the set of agents in some cycle be A_1. Then each agent i in A_1 is assigned to the house she points to (or herself), $\mu(i)$, and leaves the problem with her assignment.
Step k. In general, at the k^{th} step, let t_k be the agent with the highest order among $\widetilde{N}\setminus\cup_{n=1}^{k-1}A_n$ (according to σ), and $E_{t_k}^k=E_{t_k}^{k-1}\cup\left\{\cup_{i\in A_{k-1}}\left\{E_i^{k-1}\setminus\{\mu(i)\}\right\}\right\}$, i.e., t_k inherits all the unassigned endowments of agents in A_{k-1}. $E_i^k=E_i^{k-1}$ for $i\in\widetilde{N}\setminus\cup_{n=1}^{k-1}A_n$ and $i\neq t_k$. Then the currently unassigned agents start top trading cycle exchange with respect to E^k : each agent points to the owner of her best available option, let the set of agents in some cycle be A_k, each agent i in A_k is assigned to the house she points to ($\mu(i)$) and leaves the problem.
The algorithm terminates when all the agents are assigned, $f^{TTC}(\sigma, R_{\widetilde{N}})=\mu$.

TTC belongs to the family of hierarchical exchange rules of Pápai (2000) which are always efficient and group strategy-proof. It is also stable since each agent i is guaranteed to obtain a house weakly better than her initial endowment $\mathscr{U}(i)$.

The last case to consider is the IT structure. It is symmetric to the housing market structure and we first introduce a stable and efficient rule, the **priority-based serial dictatorship (PBSD)**. PBSD is a multiple-round serial dictatorship algorithm and starts with asking agents to pick their best available option sequentially according to some fixed ordering, but in order to satisfy stability constraint we only allow an agent to pick her best available option if she is not ranked the lowest by this option, otherwise the agent has to wait for the next round of

[13] The only modification needed is to allow an agent to be endowed with more than one house.

serial dictatorship. If at some point all the (remaining) agents are ranked the lowest by their best available option, the IT structure implies that every agent must have a different target and thus they can be assigned simultaneously. Formally, given σ, $f^{PBSD}(\sigma,\cdot)$ for a subproblem $\{\widetilde{N},\widetilde{H},\succeq_{\widetilde{H}}\}$ with the IT structure is defined as follows:

Given $R_{\widetilde{N}}$, for $S\subseteq\widetilde{H}$, $i\in\widetilde{N}$ denote $b_i(S)$ as agent i 's best choice in $S\cup\{i\}$ according to R_i.
Step 1. Denote the sequence of agents $\{\sigma(m)\}_{m=1}^{|\widetilde{N}|}$ as $\{t_m^1\}_{m=1}^n$ (so let $|\widetilde{N}|=n$) and $\widetilde{H}=\widetilde{H}_1$. If $b_i(\widetilde{H}_1)\in\mathscr{D}(i)$ for all $i\in\widetilde{N}$, let $\mu_1(i)=b_i(\widetilde{H}_1)$ for $i\in\widetilde{N}$, otherwise we use the following serial dictatorship. If $b_{t_1^1}(\widetilde{H}_1)\notin\mathscr{D}(t_1^1)$, let $\mu_1(t_1^1)=b_{t_1^1}(\widetilde{H}_1)$, otherwise let $\mu_1(t_1^1)=0$ (agent t_1^1 is tentatively assigned to a null object). Next consider t_2^1 : let $\mu_1(t_2^1)=b_{t_2^1}(\widetilde{H}_1\setminus\{\mu_1(t_1^1)\})$ if $b_{t_2^1}(\widetilde{H}_1\setminus\{\mu_1(t_1^1)\})\notin\mathscr{D}(t_2^1)$, otherwise let $\mu_1(t_2^1)=0$. Continue in this fashion till the last agent is assigned: $\mu_1(t_n^1)=b_{t_n^1}(\widetilde{H}_1\setminus\cup_{m=1}^{n-1}\{\mu_1(t_m^1)\})$ if $b_{t_n^1}(\widetilde{H}_1\setminus\cup_{m=1}^{n-1}\{\mu_1(t_m^1)\})\notin\mathscr{D}(t_n^1)$, otherwise $\mu_1(t_n^1)=0$. Let $B_1=\{i\in\widetilde{N}:\mu_1(i)\neq 0\}$.
Step *k*. In general, at the k^{th} step denote the subsequence of $\{t_m^1\}_{m=1}^n$ after agents in $\cup_{m=1}^{k-1}B_m$ are removed as $\{t_m^k\}_{m=1}^{n-|\cup_{m=1}^{k-1}B_m|}$ and $\widetilde{H}_k=\widetilde{H}\setminus\cup_{m=1}^{k-1}\mu_m(B_m)$. If $b_i(\widetilde{H}_k)\in\mathscr{D}(i)$ for all $i\in\widetilde{N}\setminus\cup_{m=1}^{k-1}B_m$, let $\mu_k(i)=b_i(\widetilde{H}_k)$ for $i\in\widetilde{N}\setminus\cup_{m=1}^{k-1}B_m$, otherwise we run the following serial dictatorship: let $\mu_k(t_1^k)=b_{t_1^k}(\widetilde{H}_k)$ if $b_{t_1^k}(\widetilde{H}_k)\notin\mathscr{D}(t_1^k)$, $\mu_k(t_1^k)=0$ otherwise, continue until the last agent $t^k_{n-|\cup_{m=1}^{k-1}B_m|}$ is assigned (for the k^{th} time). Let $B_k=\{i\in\widetilde{N}\setminus\cup_{m=1}^{k-1}B_m:\mu_k(i)\neq 0\}$.
The algorithm terminates when each agent is assigned to a house or herself, which takes at most $min\{|\widetilde{N}|,|\widetilde{H}|\}$ steps, the resulting assignment is $f^{PBSD}(\sigma,R_{\widetilde{N}})$ such that $f_i^{PBSD}(\sigma,R_{\widetilde{N}})=\mu_k(i)$ if $i\in B_k$.

Similar to the case of serial dictatorship for the house allocation problem, PBSD can give all the efficient and stable assignments for an IT structure problem:

Lemma 4 *For a subproblem $\{\widetilde{N},\widetilde{H},\succeq_{\widetilde{H}}\}$ with the IT structure, $f^{PBSD}(\sigma,\cdot)$ is stable and efficient for any σ. Moreover, for any $R_{\widetilde{N}}$, if μ is stable and efficient, there exists σ such that $f^{PBSD}(\sigma,R_{\widetilde{N}})=\mu$.*

Now there are three stable and efficient rules to deal with the three possible structures, we are ready to combine them and describe a stable and efficient rule for the problem $\{N,H,\succeq\}$ where $\succeq$ satisfies non-reversal. Given an ordering of agents σ and a preference profile R, the **priority set rule** $f^{\succeq}$ determines the allocation as following.

Step 1. Find the smallest priority set N_1 for $\{N,H,\succeq\}$. By Lemma 3, $\{N_1,H=H_1,\succeq_{H_1}\}$ can only take three possible structures. Use f^{SD} if it is the house allocation structure, f^{TTC} if it is the housing market structure, f^{PBSD} if it is the IT structure.[14] The resulting assignment is $\mu_1 : N_1 \to H_1 \cup N_1$.

Step *k*. In general, at the k^{th} step, find the smallest priority set N_k for the reduced problem $\left\{N_k = N \setminus \cup_{m=1}^{k-1} N_m, H_k = H \setminus \cup_{m=1}^{k-1} \mu_m(N_m), \succeq_{H_k}\right\}$, use f^{SD}, f^{TTC} and f^{PBSD} according to the structure of $\left\{N_k, H_k, \succeq_{H_k}\right\}$ and the resulting matching is $\mu_k : N_k \to H_k \cup N_k$.

The process terminates when every agent is assigned, which takes at most $min\{|N|, |H|\}$ steps. The resulting assignment is $f^{\succeq}(\sigma, R)$ where $f_i^{\succeq}(\sigma, R) = \mu_k(i)$ if $i \in N_k$.

Proposition 1 *Suppose* $\succeq$ *satisfies non-reversal.* $f^{\succeq}(\sigma, \cdot)$ is stable and efficient for any σ.

So we have finished the sufficiency part of the first characterization result.

Theorem 2 *Given* $\{N, H, \succeq\}$, *there exists a stable and efficient assignment for any* R *if and only if* $\succeq$ *satisfies non-reversal.*

4 Strategy-proof assignment

It can be readily seen that in general agents can manipulate PBSD thus the priority set rules are not strategy-proof. Consider an IT structure with three agents $1, 2, 3$ and three houses a, b, c where $a \in \mathscr{D}(1)$, $b \in \mathscr{D}(2)$, $c \in \mathscr{D}(3)$ and preferences are aR_1bR_11, bR_2aR_22, aR_33. If $\sigma(i) = i$ then $f_2^{PBSD}(\sigma, R) = 2$. But agent 2 would get house a by asserting this is her top choice.

Unfortunately, as shown in Theorem 3 for an IT structure with more than three agents, although an efficient and stable assignment always exists, there is no strategy-proof way of selecting such assignment. But for the case of three agents, DA with a preference-based tiebreaking rule due to Ehlers (2007) is stable, efficient and weakly group strategy-proof. Thus the priority set rules can be modified to include such DA algorithm as solution to IT structure with three agents instead of PBSD, and such modified priority set rules are weakly group strategy-proof on a smaller priority domain than the non-reversal one.

Theorem 3 *Given* $\{N, H, \succeq\}$, *the following are equivalent:*
(i) there exists a stable, efficient and strategy-proof rule,

[14] At each step, serial dictatorship, TTC and PBSD are implemented with respect to (a restriction of) σ.

(ii) there exists a stable, efficient and weakly group strategy-proof rule,

(iii) any minimal subproblem has one of the three structures: (1) house allocation, (2) housing market, (3) IT with three agents.

Finally, we consider the stronger notion of group strategy-proofness. For the special three-agent and three-house IT case, DA with the preference-based tiebreaking is group strategy-proof (Ehlers, 2007).[15] [16] As shown in Theorem 4, another impossibility result is that for an IT structure with more than three houses, any stable and efficient rule is not group strategy-proof, which suggests the "maximal domain" of priority structures is shrinking further. Recall by definition an IT structure consists of at least three houses, so if there are at least four houses and a subproblem with the IT structure exists, non-reversal implies there also exists an IT structure with four houses. Hence this structure is eliminated from any solvable problem when group strategy-proofness is required and $|H| > 3$.

Definition 5 A **weak priority reversal** consists of three distinct agents $\{i, j, k\} \subseteq N$, and some houses $\{a, b, c\} \subseteq H$ such that $\{i, j\} \succ_a k$, and $k \succ_b i, k \succ_c j$. $\succeq$ satisfies **strong non-reversal** if there is no weak priority reversal.

Ehlers and Erdil (2010) shows that the constrained efficient correspondence is efficient if and only if the priority structure satisfies *strong acyclicity*, which is defined as there are no distinct $\{i, j, k\} \subseteq N$ and $\{a, b\} \subseteq H$ such that $i \succeq_a j \succ_a k \succeq_b i$. As mentioned before, it is stronger than strong non-reversal. It can also be easily seen that under strict priorities, strong acyclicity, strong non-reversal and non-reversal are all equivalent to Ergin-acyclicity.

Strong non-reversal rules out the IT structure and implies any minimal subproblem is either a house allocation structure or a housing market structure. In light of Lemma 1, the group strategy-proofness of the priority set rules $f^{\succeq}$ for a strong non-reversal problem follows directly from the group strategy-proofness of serial dictatorship and TTC.

Theorem 4 *Given* $\{N, H, \succeq\}$, $|H| > 3$, *the following are equivalent:*

(i) there exists a stable, efficient and group strategy-proof rule,

[15]It can also be shown that in this case such DA algorithm is the only rule that satisfies stability, efficiency and group strategy-proofness.

[16] Therefore, it can be easily seen that when $|H| < 4$, non-reversal is necessary and sufficient for the existence of a stable, efficient and group strategy-proof rule.

(ii) any minimal subproblem has the house allocation or the housing market structure,
(iii) $\succeq$ satisfies strong non-reversal.

In the one-to-one matching context, some previously known problems that admit a stable, efficient and group strategy-proof rule include house allocation, housing market, house allocation with existing tenants, and also the priority-augmented allocation problem with a strict and Ergin-acyclic priority structure. They all fall into the strong non-reversal priority domain. Furthermore, it can be easily verified that $f^{\succeq}$ is reduced to serial dictatorship in house allocation, Gale's top trading cycle in housing market, the top trading cycle mechanism of Abdulkadiroğlu and Sönmez (1999) in house allocation with existing tenants, and the DA algorithm in a priority-augmented allocation problem with a strict and Ergin-acyclic priority structure.[17]

While house allocation and housing market are two classical assignment problems that admit a stable, efficient and group strategy-proof rule, Theorem 4 implies that a partial converse is also true: given any priority-augmented allocation problem (with more than three houses), such rule exists only if it can be decomposed as a sequence of subproblems defined by smallest priority sets, and each of them has either the house allocation or housing market structure. A housing market structure is not exactly a housing market problem which features equal number of houses and agents, the following corollary reinterprets Theorem 4 and provides a tighter connection to these two classical problems.

Corollary 1 *Suppose $|H| > 3$. There exists a stable, efficient and group strategy-proof rule if and only if any minimal subproblem $\{\widetilde{N}, \widetilde{H}, \succeq_{\widetilde{H}}\}$ with $|\widetilde{N}| = |\widetilde{H}|$ is either a house allocation problem or a housing market problem.*

5 Discussion

5.1 A characterization of weak non-reversal

Ehlers and Westkamp (2011) consider the same allocation problem and provide a partial characterization of the structures under which there exists a strategy-proof constrained efficient rule. They provide three necessary conditions and the first one is an acyclicity condition: A tie $i_1 \sim_o i_2$ between two distinct agents i_1, i_2 is **strongly cyclic**, if there exist agents

[17]Equivalence to DA is due to the uniqueness of the stable and efficient assignment in this case.

$j_1, j_2 \in N \setminus \{i_1, i_2\}$ and objects p_1, p_2 such that either $i_1 \succ_{p_1} j_1 \succ_o i_1$ and $i_2 \succ_{p_2} j_2 \succ_o i_2$, with $p_1 = p_2$ if $j_1 = j_2$, or $\{i_1, i_2\} \succ_{p_1} j_1 \succ_{p_2} j_2 \succ_o i_1$. A weak priority structure $\succeq$ is **EW-acyclic** if it does not contain a strongly cyclic tie. This new notion helps establish a characterization of weak non-reversal:

Proposition 2 *A priority structure $\succeq$ satisfies weak non-reversal if and only if it is Ergin-acyclic and EW-acyclic.*

Therefore, for those Ergin-acyclic but not weak non-reversal priority structures, there does not exist a stable and efficient rule, nor do they admit a strategy-proof constrained efficient rule.

5.2 The priority set rules and the hierarchical exchange rules

When any minimal subproblem has either the house allocation or housing market structure, the iterative procedure of finding the smallest priority set and applying serial dictatorship or TTC is essentially a hierarchical exchange. The hierarchical exchange rules of Pápai (2000) generalize Gale's top trading cycle algorithm by allowing endowment sets to be hierarchically determined by the **inheritance trees**. For each object $a \in H$, an inheritance tree $\Gamma_a = (V, Q)$ is a **rooted tree** where V is the set of vertices and $Q \subset V \times V$ is the set of arcs. Each vertex is labeled by an agent and each arc is labeled by an object other than a.[18] Γ_a shows how a is inherited and such inheritance can endogenously depend on previous assignments of agents. Given a list of inheritance trees $\Gamma = (\Gamma_a)_{a \in H}$, the associated hierarchical exchange rule f^Γ determines the allocation through top trading cycles with respect to Γ.

Proposition 3 *Suppose $|H| > 3$. If $\succeq$ satisfies strong non-reversal then for any σ, there exists Γ such that $f^{\succeq}(\sigma, \cdot) = f^\Gamma$. Thus there exists a stable hierarchical exchange rule if and only if $\succeq$ satisfies strong non-reversal.*

Intuitively the hierarchical exchange rules can accommodate some exogenous priorities or property rights by specifying proper inheritance trees, Proposition 3 shows formally when an exogenous priority structure can be respected. Hierarchical exchange rules are always efficient and group strategy-proof, and a stable one exists as long as there exists a stable, efficient and

[18] For ease of exposition in this subsection we assume all the objects are acceptable for each agent.

group strategy-proof rule. This result also suggests the top trading cycle procedure could have superior stability property when efficiency and incentive compatibility are the main concerns. In the same vein, Morrill (2013) shows that in the one-to-one strict priority case, Gale's top trading cycle algorithm is the "most stable" one among the class of efficient and strategy-proof rules.[19]

6 Conclusion

We considered a generalized house allocation model and searched for solvable problems in terms of stability and efficiency. When group strategy-proofness is further required, solvable problems feature a decomposition into two extensively studied allocation problems, one with social endowment and one with private endowments. The priority set technique greatly simplifies our analysis and helps establish connections to a subclass of hierarchical exchange rules. In practice, mechanisms are designed for resource allocation problems, priority structure can also be designed, and identifying solvable problems could benefit both design issues. An interesting direction for future study would be to generalize our results to the case of many-to-one matching. When multiple copies of each object are allowed and thus resources are less scarce, necessarily more priority structures become admissible. Although the decomposition result may no longer hold, similar characterizations could be established to contribute to a larger class of market design problems including school choice.

Appendix A

Proof of Lemma 2. Given $\left\{\widetilde{N}, \widetilde{H}, \succeq_{\widetilde{H}}\right\}$, by definition $\widetilde{N}$ is a priority set. For any two priority sets S_1 and S_2, $S_1 \cap S_2 \neq \phi$ otherwise there exists some $i \in S_1$, $j \in S_2$ such that $i \succ_{\succeq_{\widetilde{H}}} j$ and $j \succ_{\succeq_{\widetilde{H}}} i$, contradiction. For any $k \in S_1 \cap S_2$ and $l \notin S_1 \cap S_2$, $l \notin S_1$ or $l \notin S_2$, so $k \succ_{\succeq_{\widetilde{H}}} l$, this shows $S_1 \cap S_2$ is a priority set. □

Proof of Lemma 3. "if" part. Suppose any minimal subproblem satisfies one of the three structures. Assume to the contrary there exists a priority reversal $\{i, j\} \succ_a k \succ_b \{i, j\}$. Then

[19] Specifically, he proposes a weaker form of stability concept called *justness*, and shows that top trading cycle is the only just, efficient and strategy-proof rule. Thus any efficient and strategy-proof rule satisfies justness or a stronger stability concept is the top trading cycle algorithm.

$\left\{\widetilde{N}=\left\{i,j,k\right\},\widetilde{H}=\{a,b\},\succeq_{\{a,b\}}\right\}$ is minimal and is not a house allocation, housing market or IT structure, contradiction.

For the other direction of the proof we first show the following claim.

Claim 1 *If $\left\{\widetilde{N},\widetilde{H},\succeq_{\widetilde{H}}\right\}$ is minimal and $\succeq$ satisfies non-reversal, then (i) there does not exist $i,j\in\widetilde{N}$ such that $i\succ\succeq_{\widetilde{H}} j$, (ii) there does not exist $a\in\widetilde{H}$ and $i,j,k\in\widetilde{N}$ such that $i\succ_a j\succ_a k$.*

Proof of Claim 1. *Part* (i). Assume to the contrary, there exist agents $1,2\in\widetilde{N}$ such that $1\succ\succeq_{\widetilde{H}} 2$. Let $G_1=\left\{i\in\widetilde{N}:i\succ\succeq_{\widetilde{H}} 2\right\},G_2=\widetilde{N}\setminus G_1$, then $1\in G_1,2\in G_2$.

For any $k\in G_2$, either $k\sim_a 2$ for all $a\in\widetilde{H}$ or there exists some a such that $2\succ_a k$. If $k\sim_a 2$ for all a then $G_1\succ\succeq_{\widetilde{H}} k$. If there exists some $a\in\widetilde{H}$ such that $2\succ_a k$, then there does not exist some $i\in G_1$ and $b\in\widetilde{H}$ such that $k\succ_b i$, otherwise we have $k\succ_b\{i,2\}$ and $\{i,2\}\succ_a k$ and this is a priority reversal, so $G_1\succeq_{\widetilde{H}} k$. Then $2\succ_a k$ implies $G_1\succ_a k$, thus $G_1\succ\succeq_{\widetilde{H}} k$. This shows $G_1\succ\succeq_{\widetilde{H}} G_2$, G_1 is a strictly smaller priority set than $\widetilde{N}$, contradiction.
Part (ii). If there exist $a\in\widetilde{H}$ and $i,j,k\in\widetilde{N}$ such that $i\succ_a j\succ_a k$, then by Ergin-acyclicity $i\succ\succeq_{\widetilde{H}} k$, contradiction. □

"only if" part. Suppose $\succeq$ satisfies non-reversal. For each $a\in\widetilde{H}$, partition $\widetilde{N}$ into disjoint nonempty subsets $A_a^1,A_a^2...A_a^m$, such that $\cup_{n=1}^m A_a^n=\widetilde{N}$ and $A_a^1\succ_a A_a^2\succ_a...\succ_a A_a^m$, and $i\sim_a j$ for any $i,j\in A_a^n$. Let $A_a^1=\widetilde{N}$ if $i\sim_a j$ for all $i,j\in\widetilde{N}$. By Claim 1 $m=1$ or 2 for any $a\in\widetilde{H}$.

If for some a, $A_a^1\neq\phi$, $A_a^2\neq\phi$, then we cannot have $\left|A_a^1\right|\geqslant 2$ and $\left|A_a^2\right|\geqslant 2$. Suppose this is not true. Then without loss of generality let $\{1,2\}\subseteq A_a^1$, $\{3,4\}\subseteq A_a^2$. By Claim 1 $\exists b\in\widetilde{H}$ such that $3\succ_b 1$, so $3\in A_b^1,1\in A_b^2$, then non-reversal implies $2\in A_b^1,4\in A_b^2$. Also $\exists c$ such that $4\succ_c 2$, so $4\in A_c^1,2\in A_c^2$, then since $\{1,2\}\succ_a 4$, non-reversal implies $1\in A_c^1$. Thus $\{1,4\}\succ_c 2$ and $2\succ_b\{1,4\}$, this is a priority reversal and we reach a contradiction.

If $A_a^2=\phi$ for all $a\in\widetilde{H}$, then we have the house allocation structure.

If for some $a\in\widetilde{H}$, $A_a^2\neq\phi$ and $\left|A_a^2\right|\geqslant 2$, then $\left|A_a^1\right|=1$. Suppose $i\in A_a^1$ then $a\in\mathscr{U}(i)\neq\phi$. For any $j\neq i$, by Claim 1 there exists some b such that $j\succ_b i$ thus $j\in A_b^1,i\in A_b^2$, then non-reversal implies $\forall k\in\widetilde{N}\setminus\left\{i,j\right\}$, $k\in A_b^2$, thus $b\in\mathscr{U}(j)\neq\phi$. Now suppose for some $c\in\widetilde{H}$, $c\notin\mathscr{U}(i)$ for any $i\in\widetilde{N}$, and $c\notin I_{\widetilde{H}}$, then $\left|A_c^1\right|\geqslant 2$, $A_c^2\neq\phi$. Let $\left\{i,j\right\}\subseteq A_c^1,k\in A_c^2$, then $\left\{i,j\right\}\succ_c k$ and $k\succ_d\left\{i,j\right\}$ for some $d\in\mathscr{U}(k)$, which is a priority reversal, contradiction. Hence we have the housing market structure.

If for some $a\in\widetilde{H}$, $A_a^2\neq\phi$, and $\left|A_a^2\right|=1$ then we have the IT structure and the proof is similar to the last case. □

Proof of Lemma 4. We will use the fact that given any σ, serial dictatorship is efficient, and for any $R_{\widetilde{N}}$ and an efficient μ, there exists σ such that $f^{SD}(\sigma, R_{\widetilde{N}}) = \mu$.

First part. Given any $R_{\widetilde{N}}$ and σ, suppose the PBSD algorithm terminates at step M, so $\widetilde{N} = \cup_{k=1}^{M} B_k$. Let $B_0 = \phi$. Construct $\hat{\sigma}$ such that $\left\{\hat{\sigma}(k) : k \in \left[(\sum_{l=0}^{m-1}|B_l|)+1, \sum_{l=0}^{m}|B_l|\right]\right\} = B_m, m = 1,2,...,M$, and $\hat{\sigma}^{-1}(i) < \hat{\sigma}^{-1}(j)$ if $\sigma^{-1}(i) < \sigma^{-1}(j)$ for $i, j \in B_m, m = 1,2,...,M$. Then $f^{SD}(\hat{\sigma}, R_{\widetilde{N}}) = f^{PBSD}(\sigma, R_{\widetilde{N}})$, so $f^{PBSD}(\sigma, R_{\widetilde{N}})$ is efficient, individually rational and nonwasteful. Priorities cannot be violated since for any $i, j \in \widetilde{N}$, $j \succ_{f_i^{PBSD}(\sigma, R_{\widetilde{N}})} i$ implies $i \in B_M$ and thus j does not envy i.

Second part. Efficiency of μ implies there exists σ such that $f^{SD}(\sigma, R_{\widetilde{N}}) = \mu$. Construct $\hat{\sigma}$ in the following way. Let $\left\{\hat{\sigma}(k) : k \in \left[1, \left|\left\{i : \mu(i) \notin \mathscr{D}(i)\right\}\right|\right]\right\} = \left\{i : \mu(i) \notin \mathscr{D}(i)\right\} = B_1$.
$\left\{\hat{\sigma}(k) : k \in \left[\left|\left\{i : \mu(i) \notin \mathscr{D}(i)\right\}\right| + 1, \left|\widetilde{N}\right|\right]\right\} = \left\{i : \mu(i) \in \mathscr{D}(i)\right\} = B_2$, $\hat{\sigma}^{-1}(i) < \hat{\sigma}^{-1}(j)$ if $\sigma^{-1}(i) < \sigma^{-1}(j)$ for $i, j \in B_1$ or $i, j \in B_2$.

For any $i \in B_1$, her choice set is larger under $\hat{\sigma}$ (compared to σ) when it is her turn to choose house so $f_i^{SD}(\hat{\sigma}, R_{\widetilde{N}}) R_i \mu(i)$. Moreover, $\forall j \in B_2, B_1 \succ_{\mu(j)} j$, so stability of μ implies $\forall i \in B_1, f_i^{SD}(\hat{\sigma}, R_{\widetilde{N}}) R_i \mu(i) R_i \mu(j)$, then $f_i^{SD}(\hat{\sigma}, R_{\widetilde{N}}) \neq \mu(j)$. Clearly any agent in B_2 is not worse off under $\hat{\sigma}$ thus efficiency of μ implies $f^{SD}(\hat{\sigma}, R_{\widetilde{N}}) = \mu$. By the construction of $\hat{\sigma}$ serial dictatorship and PBSD coincide so $f^{PBSD}(\hat{\sigma}, R_{\widetilde{N}}) = f^{SD}(\hat{\sigma}, R_{\widetilde{N}}) = \mu$. □

Proof of Proposition 1. **Efficiency.** serial dictatorship, TTC and PBSD are efficient, so given R, each μ_k is efficient for $\left\{N_k, H_k, \succeq_{H_k}, R_{N_k}\right\}$, then for each k there exists $\hat{\sigma}_k : \{1,2,...,|N_k|\} \to N_k$ such that $f^{SD}(\hat{\sigma}_k, R_{N_k}) = \mu_k$. Construct $\hat{\sigma} : \{1,2,...,|N|\} \to N$ such that (i) $\hat{\sigma}^{-1}(i) < \hat{\sigma}^{-1}(j)$ if $i \in N_s, j \in N_t, s < t$; (ii) $\hat{\sigma}^{-1}(i) < \hat{\sigma}^{-1}(j)$ if $i, j \in N_k$ for some k and $\hat{\sigma}_k^{-1}(i) < \hat{\sigma}_k^{-1}(j)$. Then $f^{\succeq}(\sigma, R) = f^{SD}(\hat{\sigma}, R)$, $f^{\succeq}(\sigma, R)$ is efficient.

Stability. For any k, if $\left\{N_k, H_k, \succeq_{H_k}\right\}$ has the housing market structure, then μ_k is individually rational since agents do not point to unacceptable houses and nonwasteful since it is efficient. It respects priorities since if for some $i, j \in N_k$, $j \succ_{\mu_k(i)} i$, i either inherits $\mu_k(i)$ from j or points to j in a cycle, hence $\mu_k(j) R_j \mu_k(i)$. So combined with the stability of serial dictatorship and PBSD, each μ_k is stable for $\left\{N_k, H_k, \succeq_{H_k}, R_{N_k}\right\}$. $f^{\succeq}$ is individually rational and nonwasteful since $f^{\succeq}(\sigma, R) = f^{SD}(\hat{\sigma}, R)$. Suppose for some $i, j \in N$, $j \succ_{f_i^{\succeq}(\sigma, R)} i$, then $f_j^{\succeq}(\sigma, R) R_j f_i^{\succeq}(\sigma, R)$ if $i, j \in N_k$ for some k, by the stability of μ_k. If $i \in N_s, j \in N_t$ and $s \neq t$ then $t < s$, nonwastefulness of μ_t implies $f_j^{\succeq}(\sigma, R) R_j f_i^{\succeq}(\sigma, R)$. This shows priorities cannot be violated, hence $f^{\succeq}(\sigma, R)$ is stable. □

Proof of Theorem 2. **"if part"** follows directly from Proposition 1.

"only if" part. Suppose there exists a priority reversal $\{i,j\} \succ_a k \succ_b \{i,j\}$. Consider the following preference profile R:

$R_i : b R_i a R_i i R_i$; $R_j : b R_j a R_j j R_j$; $R_k : a R_k b R_k k R_k$; $R_l : l R_l h, \forall l \in N \setminus \{i,j,k\}, \forall h \in H$.

Suppose μ is a stable assignment, then $\mu(l) = l$ for all $l \in N \setminus \{i,j,k\}$, and by the same argument in Example 1, $\mu(k) = b$, and either $\mu(i) = a$ or $\mu(j) = a$, so μ is inefficient. Thus there does not exist a stable and efficient assignment for R. $\square$

Proof of Theorem 3. *(ii)* $\Rightarrow$ *(i)* is trivial. ***(i)*** $\Rightarrow$ ***(iii).*** Suppose there exists a stable, efficient and strategy-proof rule f, but there exists a minimal subproblem that does not have one of the three structures. Then by Theorem 2 and Lemma 3 there exists a minimal subproblem with IT structure and more than three agents. It is sufficient to consider a four-agent IT problem $\{\widetilde{N} = \{i,j,k,l\}, \widetilde{H} = \{a,b,c,d\}, \succeq_{\widetilde{H}}\}$ only, since it can be assumed that $m R_m h$ for all $m \notin \{i,j,k,l\}$, $h \in H$, then $f(m) = m$. Let $a \in \mathscr{D}(l), b \in \mathscr{D}(i), c \in \mathscr{D}(j), d \in \mathscr{D}(k)$. Construct the following preferences:

$R_l : a R_l b R_l c R_l d R_l l$, $R'_l : b R'_l a R'_l c R'_l d R'_l l$,

$R_i : b R_i c R_i a R_i i$, $R_j : c R_j b R_j d R_j j$,

$R_k : d R_k c R_k b R_k a R_k k$, $R'_k : c R'_k d R'_k b R'_k a R'_k k$.

And consider the following three preference profiles:

$R^1 = [R_l, R_i, R_j, R'_k]$

$R^2 = [R'_l, R_i, R_j, R'_k]$

$R^3 = [R'_l, R_i, R_j, R_k]$

For any σ, $(f_l^{PBSD}(\sigma, R^1), f_i^{PBSD}(\sigma, R^1), f_j^{PBSD}(\sigma, R^1), f_k^{PBSD}(\sigma, R^1)) = (d,a,b,c)$. By Lemma 4 this is the unique stable and efficient assignment for R^1, so $(f_l(R^1), f_i(R^1), f_j(R^1), f_k(R^1)) = (d,a,b,c)$. Strategy-proofness requires $f_l(R^2) = d$, then stability and efficiency implies $f(R^1) = f(R^2)$.

Now consider R^3, again there exists a unique stable and efficient assignment thus we have $(f_l(R^3), f_i(R^3), f_j(R^3), f_k(R^3)) = (b,c,d,a)$. But this shows that $f_k(R'_k, R^3_{-k}) P_k f_k(R^3)$, contradicting to f being strategy-proof.

(iii) $\Rightarrow$ ***(ii).*** It is sufficient to show there exists a stable, efficient and weakly group strategy-proof rule for an IT structure with three agents so that we can combine this rule with serial dictatorship and TTC to obtain a modified priority set rule which obviously also satisfies these three axioms. For a minimal subproblem $\{\widetilde{N}, \widetilde{H}, \succeq_{\widetilde{H}}\}$ with the IT structure and $|\widetilde{N}| = 3$, define the following ***$\mathscr{D}$-tiebreaking rule*** for the deferred acceptance algorithm with respect to an ordering of agents σ:

(i) If there is a tie between two agents i and j, and k is on the waiting list of some other house, then reject i if k is on the waiting list of some $a \in \mathscr{D}(i)$, reject j if k is on the waiting list of some $a \in \mathscr{D}(j)$, break the tie according to σ otherwise.
(ii) If the three agents apply to some $a \notin I_{\widetilde{H}}$ at the same time, reject one agent based on strict priority first and let the rejected agent apply to her next choice, then break the tie between the other two agents according to (i). Similarly, if there is a tie among three agents, reject one agent first according to σ and let the rejected agent apply to her next choice, break the tie between the other two agents according to (i).

The DA algorithm with such tiebreaking rule is denoted as $f^{DA(\mathscr{D})}(\sigma,\cdot)$. With respect to σ and a preference profile $R_{\widetilde{N}}$, define a binary relation q on the three agents: $i_1 q i_2$ if at some step of $f^{DA(\mathscr{D})}$ some house h rejects i_2 in favor of i_1 while i_3 is on the waiting list of some other house. By the $\mathscr{D}$-tiebreaking rule, it can be easily seen that q is asymmetric and ***q-acyclicity*** is satisfied: we do not have three agents j_1, j_2 and j_3 such that $j_1 q j_2 q j_3 q j_1$.

By construction $f^{DA(\mathscr{D})}$ always preserves stability. Given $R_{\widetilde{N}}$, for any $i \in \widetilde{N}$ let a_i, b_i, c_i denote agent $i's$ top three choices in $\widetilde{H} \cup \{i\}$. Let $\widetilde{N} = \left\{i, j, k\right\}$ and $f^{DA(\mathscr{D})}(\sigma, R_{\widetilde{N}}) = \mu$. We now show μ is efficient and no one has an incentive to misrepresent preference at the arbitrary preference profile $R_{\widetilde{N}}$. For simplicity we use q to denote the binary relation with respect to $R_{\widetilde{N}}$ while $\widetilde{q}$ is with respect to a manipulated preference profile under consideration.[20]

Case 1. $\left|\left\{a_i, a_j, a_k\right\}\right| = 3$. Every agent is assigned her first choice so μ is efficient and no one has an incentive to misrepresent preference.

Case 2. $a_i = a_j \neq a_k$. Without loss of generality, suppose a_i rejects j, then q-acyclicity implies $\mu(i) = a_i$ and at least one of j and k is assigned her second choice, so μ is efficient. Clearly i has no incentive to misrepresent preference. If $b_j \neq a_k$ then $\mu(k) = a_k$ and only j could potentially manipulate to obtain her first choice a_i. j cannot be assigned a_i by first applying to any house other than a_k, or first applying to a_k and getting rejected. If j first applies to a_k and $j \widetilde{q} k$ then q-acyclicity implies $\mu(j) = a_k$ so such manipulation cannot be successful.

If $b_j = a_k$ and $k q j$ then $\mu(k) = a_k$ and $\mu(j) = c_j$. j will always be rejected by her first two choices for any reported preference.

[20]Obviously $f^{DA(\mathscr{D})}$ is strategy-proof and efficient if for some $i' \in \widetilde{N}$, $a_{i'} = i'$. We restrict attention to the case where $a_{i'} \in \widetilde{H}$ for all i' in $\widetilde{N}$.

If $b_j = a_k$ and jqk by q-acyclicity $\mu(j) = b_j, \mu(k) \notin \{a_i, b_j\}$. j cannot obtain a_i no matter which house she applies to first. We now consider k's incentive. There are two potentially successful manipulation strategies for k: applying to a house $h \notin \{a_i, b_j\}$ first to change the tiebreaking between i and j (if indeed $i \sim_{a_i} j$ and iqj under true preference profile) and "pooling" with i, j: applying to a_i first. Suppose the first manipulation strategy is successful, then $i \sim_{a_i} j$, and $j\tilde{q}i, i\tilde{q}k$, thus $h \in I_{\tilde{H}} \cup \mathscr{D}(k)$, and $\sigma^{-1}(j) < \sigma^{-1}(i)$. From iqj we have $b_j \in \mathscr{D}(j)$, but this contradicts to jqk. Now assume the second manipulation strategy is successful, all the three agents apply to a_i first, k cannot be the first rejected agent. If i is the first rejected agent, then $a_i \in I_{\tilde{H}}, \sigma^{-1}(j) < \sigma^{-1}(i)$. So iqj implies $b_j \in \mathscr{D}(j)$, contradicting to jqk. If j is the first rejected agent, i must be the second rejected agent, this implies $i \sim_{a_i} k$ and $b_j \notin \mathscr{D}(k)$. Then jqk implies $j \sim_{b_j} k$, and $a_i \notin \mathscr{D}(j)$, then $a_i \in I_{\tilde{H}}$, thus $\sigma^{-1}(j) < \sigma^{-1}(k)$, contradicting to jqk.

Case 3. $a_i = a_j = a_k$. Without loss of generality suppose k is the first rejected agent and j is the second rejected agent, then $\mu(i) = i$, at least one of j and k is assigned her second choice thus μ is efficient. q-acyclicity implies j and k can never obtain a_i by applying to any house first. So we are only left to show if j or k is assigned her third choice (so $b_j = b_k$) then she cannot manipulate to obtain second choice. If $\mu(k) = c_k$, the only potential manipulation strategy for k is to apply to some house $h \notin \{a_i, b_k\}$ first to change the tiebreaking between i and j, and by the same argument in case 2 such strategy cannot be successful. If $\mu(j) = c_j$, similarly the only possible manipulation for j is through applying to some house $h \neq a_i$ first to change the tiebreaking between i and k. If such manipulation is successful then $a_i \in I_{\tilde{H}}$ and $\sigma^{-1}(i) < \sigma^{-1}(k), \sigma^{-1}(j) < \sigma^{-1}(k)$, thus $h \in \mathscr{D}(i)$ and then j is assigned $h = b_j \in \mathscr{D}(i)$, contradicting to kqj.

Finally we show $f^{DA(\mathscr{D})}$ is weakly group strategy-proof. It is enough to consider the case $a_i = a_j$ and iqj, jqk so $\mu(i) = a_i, \mu(j) = b_j, \mu(k) \in \{b_k, c_k\}$. So either $a_k \neq a_i$ or $a_k = a_i$ and k is the first rejected agent. Suppose j and k can jointly manipulate such that both are strictly better-off, then j is assigned a_i, so $i \sim_{a_i} j$ and $\sigma^{-1}(i) < \sigma^{-1}(j)$. q-acyclicity implies j must apply to a_i first. For j to be assigned a_i,k can either apply to a_i first, be the first rejected agent and apply to $h \in \mathscr{D}(i)$ next to influence the tiebreaking between i and j, or by simply applying to h first. But under such joint manipulation k is assigned h and cannot be strictly better-off. □

Proof of Theorem 4. ***(iii)*** $\Rightarrow$ ***(i).*** When $\succeq$ satisfies strong non-reversal, any minimal subproblem has either the house allocation structure or the housing market structure. Both serial dictatorship and TTC are strategy-proof and nonbossy, it follows that $f^{\succeq}$ is strategy-proof and nonbossy, hence group strategy-proof by Lemma 1. $f^{\succeq}$ is also stable and efficient by Proposition 1.

***(ii)* ⇒ *(iii)*.** Given (ii), by Lemma 3 $\succeq$ satisfies non-reversal. Suppose $\succeq$ does not satisfy strong non-reversal, then there exists three distinct agents and three distinct houses such that $\{i,j\} \succ_a k, k \succ_b i, k \succ_c j$. Non-reversal implies $\{\widetilde{N} = \{i,j,k\}, \widetilde{H} = \{a,b,c\}, \succeq_{\widetilde{H}}\}$ is a minimal subproblem with the IT structure, contradiction.

***(i)* ⇒ *(ii)*.** Suppose there exists a stable, efficient and group strategy-proof rule f. Assume (ii) is not true, then by Theorem 3 there exists a minimal subproblem $\{\widetilde{N} = \{1,2,3\}, \widetilde{H} = \{a,b,c\}, \succeq_{\widetilde{H}}\}$ with the IT structure and three agents. Suppose $a \in \mathscr{D}(1)$, $b \in \mathscr{D}(2)$, $c \in \mathscr{D}(3)$. Since $|H| > 3$, there exists some $d \notin \{a,b,c\}$ and $\{\widetilde{N} = \{1,2,3\}, \widetilde{H} = \{a,b,c,d\}, \succeq_{\widetilde{H}}\}$ must be also a minimal subproblem with the IT structure. It is sufficient to consider this subproblem only. There are two cases:

Case 1. $d \in \mathscr{D}(i)$ for some i. Without loss of generality, suppose $d \in \mathscr{D}(3)$.

The following result, from Lemma 1 of Svensson (1999), will be helpful.

Claim 2 *(Svensson, 1999). A rule f is group strategy-proof if and only if for any two preference profiles R, R' such that for any $i \in N, a \in H \cup \{i\}$, $f_i(R) R_i a$ implies $f_i(R) R'_i a$, then $f(R) = f(R')$.*

By Lemma 4, $f(c,d \mid c,d \mid d,a) \in \{(c,d,a),(d,c,a)\}$.[21] First we want to show $f(c,d \mid c,d \mid d,a) = (d,c,a)$. Again, by Lemma 4 $f(a,c,b \mid c,a \mid c,a) = (b,c,a)$, by Claim 2 $f(c,a,b \mid c,d \mid c,a) = (b,c,a)$. So if $f(c,d \mid c,d \mid d,a) = (c,d,a)$, then given true preference profile $(c,a,b \mid c,a \mid c,a)$ agent 1 and 3 can jointly manipulate and agent 1 will be strictly better-off. Thus $f(c,d \mid c,d \mid d,a) = (d,c,a)$. By a symmetric argument, we can show $f(c,d \mid c,d \mid d,b) = (c,d,b)$.

Given $f(c,d \mid c,d \mid d,a) = (d,c,a)$, strategy-proofness and efficiency imply $f(c,1 \mid c,d \mid d,a) = (1,c,d)$, then by Claim 2 $f(c,1 \mid c,2 \mid d,3) = (1,c,d)$. By a symmetric argument we can show $f(c,d \mid c,d \mid d,b) = (c,d,b)$ implies $f(c,1 \mid c,2 \mid d,3) = (c,2,d)$, contradiction.

Case 2. $d \in I_{\widetilde{H}}$, i.e., $1 \sim_d 2 \sim_d 3$.

By Lemma 4, $f(a,d,b \mid d,a \mid d,a) \in \{(b,d,a),(b,a,d)\}$, then strategy-proofness and efficiency imply $f(d,a,b \mid d,a \mid d,a) \in \{(b,d,a),(b,a,d)\}$.

Step 1. *$f(d,a,b \mid d,a \mid d,a) = (b,d,a)$ implies $f(d,b \mid d,b,a \mid d,b) = (d,a,b)$.*

[21] For simplicity we denote $f(R_1 : cR_1d; R_2 : cR_2d; R_3 : dR_3a)$ as $f(c,d \mid c,d \mid d,a)$, and irrelevant preference rankings of houses are not listed.

By Claim 2, $f(|\ d,a,b \mid d,a \mid d,a) = (b,d,a) \Rightarrow f(d,b,a \mid d,b,a \mid d,a,b) = (b,d,a)$, then strategy-proofness implies $f_3(d,b,a \mid d,b,a \mid d,b,a) \neq d$. Since by Lemma 4 $f_2(d,b,a \mid b,d,a \mid d,b,a) = a$, then $f_2(d,b,a \mid d,b,a \mid d,b,a) = a$, combined with $f_3(d,b,a \mid d,b,a \mid d,b,a) \neq d$ we have $f(d,b,a \mid d,b,a \mid d,b,a) = (d,a,b)$ by efficiency. It follows that $f(d,b \mid d,b,a \mid d,b) = (d,a,b)$.

Step 2. *$f(d,a,b \mid d,a \mid d,a) = (b,d,a)$ is not possible.*

Assume to the contrary $f(d,a,b \mid d,a \mid d,a) = (b,d,a)$, then by Claim 2 $f(d,a,b \mid d,c \mid d,a) = (b,d,a)$, then $f_3(d,a,b \mid d,c \mid d,c,a) \in \{c,a\}$. If $f_3(d,a,b \mid d,c \mid d,c,a) = a$, then nonbossiness implies $f(d,a,b \mid d,c \mid d,c,a) = (b,d,a)$, nonwastefulness is violated. Thus $f_3(d,a,b \mid d,c,2 \mid d,c,a) = c$. Since $2 \succ_c 3$, stability implies $f_2(d,a,b \mid d,c \mid d,c,a) = d$. So we have $f(d,a,b \mid d,c \mid d,c,a) = (a,d,c)$, then clearly $f(d,a,b \mid d,b \mid d,c,a) = (a,d,c)$.

By step 1 $f(d,b \mid d,b,a \mid d,b) = (d,a,b)$. By a similar argument, $f(d,b \mid d,b,a \mid d,b) = (d,a,b) \Rightarrow f(d,c \mid d,b,a \mid d,b) = (d,a,b) \Rightarrow f(d,c \mid d,b,a \mid d,c,b) = (d,b,c) \Rightarrow f(d,a,b \mid d,b,a \mid d,c,b) = (d,b,c) \Rightarrow f(d,a,b \mid d,b \mid d,c,a) = (d,b,c)$, contradiction.

Step 3. *$f(d,a,b \mid d,a \mid d,a) = (b,a,d)$, $f(d,b \mid d,b,c \mid d,b) = (d,c,b)$, $f(d,c \mid d,c \mid d,c,a) = (c,d,a)$.*[22]

Recall from the beginning $f(d,a,b \mid d,a \mid d,a) \in \{(b,d,a),(b,a,d)\}$. Step 2 implies $f(d,a,b \mid d,a \mid d,a) = (b,a,d)$. By symmetry, we also have $f(d,b \mid d,b,c \mid d,b) = (d,c,b)$, $f(d,c \mid d,c \mid d,c,a) = (c,d,a)$.

Step 4. *Such stable, efficient and group strategy-proof rule f does not exist.*

Consider the following preference profile R. $R_1 : d,b,a; R_2 : d,c,b; R_3 : d,a,c$. By step 3, $f(d,c \mid d,c \mid d,c,a) = (c,d,a)$, thus by Claim 2 $f(d,c \mid d,c,b \mid d,a,c) = (c,d,a)$, then $f_1(R') \neq d$ otherwise agent 1 can manipulate and be strictly better-off. Similarly, $f(d,a,b \mid d,a \mid d,a) = (b,a,d) \Rightarrow f(d,b,a \mid d,a \mid d,a,c) = (b,a,d) \Rightarrow f_2(R) \neq d$. And $f(d,b \mid d,b,c \mid d,b) = (d,c,b) \Rightarrow f(d,b,a \mid d,c,b \mid d,b) = (d,c,b) \Rightarrow f_3(R) \neq d$. Nonwastefulness is violated hence such rule f does not exist. $\square$

Proof of Corollary 1. "only if" part follows directly from Theorem 4.

"if" part. We show that if any minimal subproblem with $\left|\widetilde{N}\right| = \left|\widetilde{H}\right|$ is either a house allocation problem or a housing market problem then strong non-reversal is satisfied. Assume to the

[22] The idea is similar to the preference-based tiebreaking rule in the proof of Theorem 3. When all the agents have the same first choice d and same second choice $h \in \{a,b,c\}$, if $h \in \mathscr{D}(i)$ then i will be assigned her third choice, and who gets d depends on i's third choice: if i's third choice belongs to $\mathscr{D}(j)$ $(j \neq i)$ then the third agents k will be assigned d.

contrary there exists a weak priority reversal $\{i, j\} \succ_a k, k \succ_b i, k \succ_c j$. If $b = c$, pick any $h \in H \backslash \{a, b\}$ and $\{\widetilde{N} = \{i, j, k\}, \widetilde{H} = \{a, b, h\}, \succeq_{\widetilde{H}}\}$ is minimal, but it is not a house allocation problem or a housing market problem, contradiction. If $b \neq c$, $\{\widetilde{N} = \{i, j, k\}, \widetilde{H} = \{a, b, c\}, \succeq_{\widetilde{H}}\}$ is minimal and a similar contradiction will be reached. □

Proof of Proposition 2. "if" part. Suppose $\succeq$ does not satisfy weak non-reversal, then there exists distinct $i, j, k \in N$ and $a, b \in H$ such that $\{i, j\} \succ_a k \succ_b \{i, j\}$. If $i \sim_b j$ then this is a strongly cyclic tie, thus $\succeq$ is not EW-acyclic. If $i \succ_b j$ there is an Ergin-cycle $k \succ_b i \succ_b j \succ_a k$, if $j \succ_b i$ there is an Ergin-cycle $k \succ_b j \succ_b i \succ_a k$, so $\succeq$ is not Ergin-acyclic when it is not $i \sim_b j$.
"only if" part. We have shown weak non-reversal implies Ergin-acyclicity. Suppose $\succeq$ satisfies weak non-reversal but assume to the contrary there exists a strongly cyclic tie $i_1 \sim_o i_2$ between two distinct agents i_1, i_2, then there exists $j_1, j_2 \in N \setminus \{i_1, i_2\}$ and $p_1, p_2 \in H$ such that we have four possible cases:
Case 1. $j_1 \neq j_2, p_1 \neq p_2$ and $\{j_1, j_2\} \succ_o i_1 \sim_o i_2, i_1 \succ_{p_1} j_1, i_2 \succ_{p_2} j_2$. Then $j_2 \succeq_{p_1} i_1$ otherwise $\{j_1, j_2\} \succ_o i_1 \succ_{p_1} \{j_1, j_2\}$ which is a strong priority reversal. Similarly $j_1 \succeq_{p_1} i_2$ otherwise $j_1 \succ_o \{i_1, i_2\} \succ_{p_1} j_1$, and $j_2 \succeq_{p_2} i_1$ otherwise $j_2 \succ_o \{i_1, i_2\} \succ_{p_2} j_2$. But now we have $\{i_1, j_2\} \succ_{p_1} i_2 \succ_{p_2} \{i_1, j_2\}$, contradiction.
Case 2. $j_1 \neq j_2, p_1 = p_2$ and $\{j_1, j_2\} \succ_o i_1 \sim_o i_2, i_1 \succ_{p_1} j_1, i_2 \succ_{p_1} j_2$. By the same argument in case 1 we have $j_2 \succeq_{p_1} i_1$. Then $\{i_1, i_2\} \succ_{p_1} j_1 \succ_o \{i_1, i_2\}$, contradiction.
Case 3. $j_1 = j_2, p_1 = p_2$ and $j_1 \succ_o \{i_1, i_2\} \succ_{p_1} j_1$, contradiction.
Case 4. $\{i_1, i_2\} \succ_{p_1} j_1, j_1 \succ_{p_2} j_2, j_2 \succ_o i_1 \sim_o i_2$. First it can be easily verified that $\left|\{o, p_1, p_2\}\right| = 3$ otherwise there always exists an Ergin-cycle. Then $j_2 \succ_{p_1} j_1$ otherwise $\{i_1, i_2\} \succ_{p_1} j_2 \succ_o \{i_1, i_2\}$. Similarly $j_2 \succ_o j_1$ otherwise $j_1 \succ_o \{i_1, i_2\} \succ_{p_1} j_1$, and $i_1 \succeq_{p_2} j_1$ otherwise $j_1 \succ_{p_2} \{i_1, j_2\} \succ_{p_1} j_1$. But now we have $\{i_1, j_1\} \succ_{p_2} j_2 \succ_o \{i_1, j_1\}$, contradiction. □

Proof of Proposition 3. When $\succeq$ satisfies strong non-reversal, given σ we construct Γ such that $f^{\Gamma} = f^{\succeq}(\sigma, \cdot)$. We first define some useful concepts. For $a \in H$, $T_a = (V, Q)$ is a **rooted tree** for a where V is the set of vertices and $Q \subset V \times V$ is the set of arcs. Each vertex $v \in V$ is labeled by an agent $\mathscr{L}(v) \in N$ and each arc (v_i, v_j) is labeled by a house $\mathscr{H}(v_i, v_j) \in H \setminus \{a\}$. v_0 is the **root** of T_a if it is the unique vertex such that there is no $v \in V$ with $(v, v_0) \in Q$. A **terminal vertex** is a vertex v such that there is no $v' \in V$ with $(v, v') \in Q$. A ***Q*-path** from v_1 to v_r is a sequence $\{v_s\}_{s=1}^{r}$ where $r \geqslant 2$, such that for all $s = 1, ..., r-1$, $(v_s, v_{s+1}) \in Q$. If there exists a Q-path $\{v_s\}_{s=0}^{r}$ from v_0 to v_r, then denote $\mathscr{O}(v_0, v_r \mid T_a) = \cup_{s=0}^{r-1} \{\mathscr{H}(v_s, v_{s+1})\}$ and $\mathscr{A}(v_0, v_r \mid T_a) = \cup_{s=0}^{r-1} \{\mathscr{L}(v_s)\}$. Also let $\mathscr{O}(v_0, v_0 \mid T_a) = \mathscr{A}(v_0, v_0 \mid T_a) = \phi$. For a subproblem $\{\widetilde{N}, \widetilde{H}, \succeq_{\widetilde{H}}\}$ with the smallest priority set A, $i \in A$ is the **top priority agent** for $\left\{\widetilde{N}, \widetilde{H}, \succeq_{\widetilde{H}}\right\}$ if *(i)* $\left\{A, \widetilde{H}, \succeq_{\widetilde{H}}\right\}$ has the house allocation structure and $\sigma^{-1}(i) < \sigma^{-1}(j)$ for all $j \in A \setminus \{i\}$, or *(ii)* it has the housing market structure, $a \in I_{\widetilde{H}}$ and $\sigma^{-1}(i) < \sigma^{-1}(j)$ for all $j \in A \setminus \{i\}$, or *(iii)* it has the housing market structure and $a \in \mathscr{U}(i)$. $v \in V$ is a **top priority vertex** if $v = v_0$, or there is a Q-path from v_0 to v and when $N \setminus \{\mathscr{A}(v_0, v \mid T_a) \cup \{\mathscr{L}(v)\}\} \neq \phi$, $H \setminus \{\mathscr{O}(v_0, v \mid T_a) \cup \{a\}\} \neq \phi$, $\mathscr{L}(v)$ is the top priority agent for $\left\{N \setminus \mathscr{A}(v_0, v \mid T_a), H \setminus \mathscr{O}(v_0, v \mid T_a), \succeq_{H \setminus \mathscr{O}(v_0, v|T_a)}\right\}$. Assume properties $(A.1)$, $(A.2)$, $(B.1)$, $(B.2)$, $(B.3)$ and $(C.1)$ of a rooted tree in Pápai (2000) hold.

Let ξ be the set of rooted trees for a such that each terminal vertex is a top priority vertex. Consider the rooted tree $T_a^0 = (V = \{v_0\}, Q = \phi)$, where $\mathscr{L}(v_0)$ is the top priority agent for $\{N, H, \succeq_H\}$, clearly $T_a^0 \in \xi$. So a belongs to the initial endowment set of agent $\mathscr{L}(v_0)$. We now define a function $\varphi : \xi \to \xi$ that expands a rooted tree for a such that repeated application of φ on T_a^0 will lead to a full inheritance tree Γ_a. Given $T_a = (V, Q) \in \xi$, denote the set of terminal vertices as $\tau(T_a)$. $\forall\, t \in \tau(T_a)$, if $N = \mathscr{A}(v_0, t \mid T_a) \cup \{\mathscr{L}(t)\}$ or $H = \mathscr{O}(v_0, t \mid T_a) \cup \{a\}$, let $V^t = Q^t = \phi$ (no expansion is needed from t). Otherwise let A^t be the smallest priority set of $\left\{N \setminus \mathscr{A}(v_0, t \mid T_a), H \setminus \mathscr{O}(v_0, t \mid T_a), \succeq_{H \setminus \mathscr{O}(v_0, t|T_a)}\right\}$. If $A^t = \{\mathscr{L}(t)\}$, we go to the endogenous endowment stage directly. Otherwise list agents in $A^t \setminus \{\mathscr{L}(t)\}$ according to σ as $\{x_s\}_{s=1}^{|A^t|-1}$.

Fixed endowment stage. We first expand a rooted tree by letting a be inherited to other agents in A^t following σ. Construct vertex $v_h(t)$ for each $h \in H \setminus \{\mathscr{O}(v_0, t \mid T_a) \cup \{a\}\}$.
Let $V_1^t = \{v_h(t)\}_{h \in H \setminus \{\mathscr{O}(v_0, t|T_a) \cup \{a\}\}}$, and $\mathscr{L}(v) = x_1$ for all $v \in V_1^t$. $Q_1^t = \{(t, v_h(t))\}_{h \in H \setminus \{\mathscr{O}(v_0, t|T_a) \cup \{a\}\}}$ and $\mathscr{H}(t, v_h(t)) = h$. Then $T_1^t = (V \cup V_1^t, Q \cup Q_1^t)$. In general, given V_k^t, Q_k^t, T_k^t, define V_{k+1}^t and Q_{k+1}^t as follows: for $\tilde{v} \in V_k^t$, construct $v_h(\tilde{v})$ for each $h \in H \setminus \left\{\mathscr{O}(v_0, \tilde{v} \mid T_k^t) \cup \{a\}\right\}$.
Then $V_{k+1}^t = \cup_{\tilde{v} \in V_k^t} \left\{\{v_h(\tilde{v})\}_{h \in H \setminus \left\{\mathscr{O}(v_0, \tilde{v}|T_k^t) \cup \{a\}\right\}}\right\}$ and $\mathscr{L}(v) = x_{k+1}$ for all $v \in V_{k+1}^t$.
$Q_{k+1}^t = \cup_{\tilde{v} \in V_k^t} \left\{\{(\tilde{v}, v_h(\tilde{v}))\}_{h \in H \setminus \left\{\mathscr{O}(v_0, \tilde{v}|T_k^t) \cup \{a\}\right\}}\right\}$ and $\mathscr{H}((\tilde{v}, v_h(\tilde{v})) = h$. $T_{k+1}^t = (V \cup \left\{\cup_{s=1}^{k+1} V_s^t\right\}, Q \cup \left\{\cup_{s=1}^{k+1} Q_s^t\right\})$. After $V_{|A^t|-1}^t$, $Q_{|A^t|-1}^t$ and $T_{|A^t|-1}^t$ are constructed, we go to the next stage.

Endogenous endowment stage. If $H = \mathscr{O}(v_0, v \mid T_{|A^t|-1}^t) \cup \{a\}$ for some $v \in V_{|A^t|-1}^t$ or $N = \mathscr{A}(v_0, t \mid T_a) \cup A^t$, let $V_{|A^t|}^t = V_{|A^t|-1}^t$, $Q_{|A^t|}^t = Q_{|A^t|-1}^t$. Otherwise we let a be inherited to some agent in the next smallest priority set and clearly such inheritance depends on the previous assignments. Construct $Q_{|A^t|}^t$ and $V_{|A^t|}^t$ as in the last stage except the labeling of vertices in $V_{|A^t|}^t$. For any $v \in V_{|A^t|}^t$, let $\mathscr{L}(v)$ be the top priority agent for $\left\{N \setminus \mathscr{A}(v_0, v \mid T_{|A^t|-1}^t), H \setminus \mathscr{O}(v_0, v \mid T_{|A^t|-1}^t), \succeq_{H \setminus \mathscr{O}(v_0, v|T_{|A^t|-1}^t)}\right\}$.
Then $V^t = \cup_{s=1}^{|A^t|} V_s^t$, $Q^t = \cup_{s=1}^{|A^t|} Q_s^t$.

Let $V' = V \cup \left\{\cup_{t \in \tau(T_a)} V^t\right\}$, $Q' = Q \cup \left\{\cup_{t \in \tau(T_a)} Q^t\right\}$. Then $\varphi(T_a = (V, Q)) = T_a' = (V', Q') \in \xi$. Repeated applications of φ on T_a^0 give us an inheritance tree for a: there exists some integer n

such that $\varphi^n(T_a^0) = \varphi^{n+1}(T_a^0) = \Gamma_a$ where Γ_a satisfies properties $(C.2)$ and $(C.3)$ in Pápai (2000). After Γ_a is specified for all $a \in H$, by the construction $f^\Gamma = f^{\succeq}(\sigma, \cdot)$. □

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