Optimal consumption under uncertainty, liquidity constraints, and bounded rationality*

Ömer Özak†
Department of Economics
Southern Methodist University

May 15, 2012

Abstract

I study how boundedly rational agents can learn the solution to an infinite horizon optimal consumption problem under uncertainty and liquidity constraints. I present conditions for the existence of an optimal linear consumption rule and characterize it. Additionally, I use an empirically plausible theory of learning to generate a class of adaptive learning algorithms that converges to the optimal rule. This provides an adaptive and boundedly rational foundation to neoclassical consumption theory.

Key Words: Adaptive learning models, bounded rationality, dynamic programming, consumption function, behavioral economics, liquidity constraint, Markov process
JEL classification: C6, D8, D9, E21

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*I wish to thank Peter Howitt for his continuous support, guidance, and encouragement. Also, I wish to thank the participants at the Econometric Society's LAMES 2008 and NASM 2009 meetings, and especially Kfir Eliaz, Glenn Loury, Tim Salmon, and David Weil, for comments and helpful discussions on previous versions of the paper.

†Email: `omer@omerozak.com`. Send correspondence to: PO Box 0496, Southern Methodist University, Dallas, TX 75275-0496, USA.

1 Introduction

Rationality is one of the main tenets of modern economics and though it has proven fruitful in all areas of economics, it has recently been subject to attacks both on theoretical and empirical grounds (see for example the section Anomalies in the Journal of Economic Perspectives). One line of criticism argues that the solution to certain problems under realistic settings is too complex for agents to solve. For example, Arthur (1994) argues that "beyond a certain level of complexity human logical capacity ceases to cope". Similarly, Conlisk (1996) in his survey on the use of bounded rationality in economics, admits that "there are critical physiological limits on human cognition"; and according to Simon (1990) "[b]ecause of the limits on their computing speeds and power, intelligent systems must use approximate methods to handle most tasks. Their rationality is bounded."

The modern theory of consumption under liquidity constraints and uncertainty, which is one of the main building blocks of many macroeconomic models, is one area that clearly is subject to this criticism. For example, Carroll (2001) presents this theory and argues that "when there is uncertainty about the future level of labor income, it appears to be impossible under plausible assumptions about the utility function to derive an explicit solution for consumption as a direct (analytical) function of the model's parameters". Similarly, Allen and Carroll (2001) admit that "finding the exact nonlinear consumption policy rule (as economists have done) is an extraordinarily difficult mathematical problem".

In order to answer this line of critiques, economists have tried to provide bounded rationality foundations to optimal behavior, especially within game theory (Fudenberg and Levine, 1998), and macroeconomics (Evans and Honkapohja, 2001; Sargent, 1993). Still, the study of how agents learn the optimal policy to an infinite horizon dynamic programming problem under uncertainty, and the consumption function in particular, has largely been ignored or has generated negative results (Allen and Carroll, 2001; Lettau and Uhlig, 1999).[1] In contrast to the negative theoretical results, empirical and experimental evidence suggests that agents *do learn to behave as if* they had solved the optimal consumption problem (see Brown, Chua, and Camerer, 2009, and references therein).

In this paper I study an infinite horizon optimal consumption problem under uncertainty, liquidity constraints, and bounded rationality. I follow the previous literature in assuming that boundedly rational agents use a consumption rule that is linear in wealth.[2] I provide conditions for the existence of a unique optimal linear consumption rule for a class of consumption problems. Additionally, I show that the optimal linear consumption rule is learnable if agents use an adaptive learning mechanism similar to the one proposed in the numerical exercise of Howitt and Özak (2009).

[1] There is a large literature which studies dynamic programming problems in which agents do not hold Rational Expectations, but are otherwise fully rational. The objective of this literature is to understand the conditions under which the expectation mechanism held by agents converges to Rational Expectations (Branch, Evans, and McGough, 2010; Sargent, 1993). This is not the problem I am alluding to here. In this setting agents are not able or willing to solve the optimal consumption problem, even if they had the correct expectational mechanism.

[2] Gabaix (2011) has suggested that boundedly rational agents only use “sparse” rules of behavior. In this case, the assumption is that agents focus only on wealth and disregard all other variables. As can be seen from the results and proofs below, they can be extended to include linearity in other variables, without affecting the results.

The approach to learning that I study, which I call the HO-algorithm, is based on Euler-equations, where agents change their linear consumption rule in response to differences between the marginal utility implied by the rule and the discounted marginal utility of next period's consumption under the rule. This approach is close in spirit to "learning direction theory" (Selten and Buchta, 1999; Selten and Stoecker, 1986), which has been proposed as an explanation for behavior observed in various experimental settings. According to that theory, an agent's success or failure changes her behavior in the direction that increases her expected payoff in the following opportunity she has for action. In the H-O algorithm, agents adjust their consumption rule if it failed to equalize the marginal utilities of consumption between yesterday and today in a way that agents regret. While both learning direction theory and the H-O algorithm explain the direction of change, the H-O algorithm also tells agents by how much they ought to change their behavior.

My approach differs from the one used by Lettau and Uhlig (1999) and Allen and Carroll (2001), who use the accumulated performance of a rule as measured by the discounted sum of utilities as a base for their learning mechanisms. In these papers, agents estimate the value function of their respective problem in order to select the best rule. In particular, Lettau and Uhlig (1999) use a setup similar to equation (3.3a) below, while Allen and Carroll (2001) have a setup similar to equation (EV^b).

Lettau and Uhlig (1999) assume that agents use an adaptive method to update their estimates, which are the input of the classifier system they study. Using methods of stochastic approximation, they characterize the set of rules that will be learnt by agents, as long as these rules are learnable. In particular, they show that agents might not learn to use the rational rule, even if it is one of the rules the agents are endowed with. Their analysis has two drawbacks, besides the requirement of a finite set of states, which can be overcome applying theorem 2.2 below. First, their approach requires the set of rules to be finite. Second, they cannot determine welfare properties of the rules that are learnt, especially when the rational rule is not available or if the rational rule is not equivalent to a mix of the available rules.

Allen and Carroll (2001) assume agents learn by studying a finite family of linear consumption rules and choosing the best among them. In particular, they assume agents select a rule by making consumption choices using each rule for many periods until they have an accurate estimator of the value function for each rule and each initial wealth level. This is tantamount to assuming agents perform something similar to a Monte Carlo simulation. They show numerically that for their parameters there is a linear rule that is quite close to the rational rule in terms of equivalent consumption. Additionally, their simulations show that the learning mechanism, chooses the "best" rule among the ones being studied by the agent. The major problem with their approach is that their learning procedure requires too much time (*4 million periods!*) to get close to the optimal rule, and thus "is not an adequate description of the process by which consumers learn about consumer behavior" (Allen and Carroll, 2001, p.268).

In their numerical simulations, Howitt and Özak (2009) show that for certain parameter configurations, agent's welfare losses are low from following a simple linear consumption rule, and that if agents use the HO-algorithm they can learn with high probability a linear consumption rule that is almost optimal in less than 500 periods.

My analytical results generalize the numerical ones found by Howitt and Özak (2009) and solve some of the problems raised in the literature. In particular, I present conditions for the existence of an optimal linear consumption rule under liquidity constraints and bounded rationality and characterize it. Additionally, I show that the HO-algorithm converges to the stationary points of a particular ordinary differential equation (ODE). This implies that applying the HO-algorithm to *any* linear consumption rule in an uncountable and compact set, for different levels of risk-aversion or impatience, with high probability will converge to the stationary points of this ODE. Clearly, if the ODE has a globally asymptotically stable stationary point, then *every* linear consumption will converge to it with probability one. Finally, I study the relation between the optimal linear consumption rule and the stationary points of the ODE. I find that generally they will be "close" to each other and will coincide for a wide class of income processes. This seems to be the first positive answer to the question "can boundedly rational agents learn to consume optimally?".

There are various reasons why H-O algorithm seems like a good candidate for behavior under bounded rationality. First, as mentioned above, it is similar to learning direction theory, which gives it empirical relevance. Second, it allows the set of rules and states to be uncountable, while keeping the informational and computational requirements for agents very low. On the contrary, both requirements are increasing in the number of rules and states in the previous literature.

The paper proceeds as follows: Section 2 presents the model and proves a new theorem on existence and uniqueness of invariant ergodic distributions for certain Markovian dynamics; section 3 presents the family of consumption rules to be studied and presents the conditions for the existence of an optimal linear consumption function, section 4 presents the learning algorithm and its properties; and section 5 concludes. All technical proofs are left for the appendix.

2 The model

Time evolves discretely and is indexed by t. Agents are infinitely lived and born with an initial wealth $w_0 > 0$. Every period they consume an amount c_t out of their current wealth w_t, receive interest on their savings and before taking the next consumption decision, get an income y_{t+1}, so that their wealth evolves according to

$$w_{t+1} = R_{t+1}\big(w_t - c_t\big) + y_{t+1}. \tag{2.1}$$

Agents are liquidity constrained, which implies that their current consumption is given by

$$c_t = c(w_t) = \min\{w_t, \hat{c}(w_t)\} \tag{2.2}$$

where $c(w)$ is a function that determines for each wealth level $w \geq 0$ the amount to be consumed. Consumption gives an agent a per period level of utility $u(c)$, where $u(\cdot)$ is continuous, strictly increasing, concave, twice continuously differentiable, and such that for some $K > 0$ and[3] $\phi : \mathbb{R}_+ \to \mathbb{R}_{++}$

$$\|u\|_\phi = \sup \frac{|u(c)|}{\phi(c)} < K, \quad \text{and} \quad \left\|u^{(n)}\right\|_\phi = \sup \frac{\left|\frac{d^n u(c)}{dc^n}\right|}{\phi(c)} < K, \text{for all } n \leq n^*,\ n^* \geq 3, \tag{2.3}$$

where $\phi(\cdot)$ is a continuous function for which the level sets $C_\phi = \{w \in \mathbb{R}_+ \mid \phi(w) \leq d\}$ for some $d \in \mathbb{R}_+$ are compact. She discounts her per period utility at a rate $\beta \in (0, 1)$ and gives her a lifetime utility level $U = \sum_{t=1}^{\infty} \beta^t u(c_t)$.

Assumption A. The function $\hat{c}(w)$ in (2.2) is strictly increasing and concave.

Below I will give specific forms to this function, but for now I can leave it at this general level, since some results do not depend on the specifics of this consumption policy.

Lemma 2.1. *If there exists $\delta > 0$ such that $\hat{c}(w) - w > 0$ for all $w \in (0, \delta)$ and either (i) $\lim_{w\to\infty} \hat{c}(w) - w = -\infty$ or (ii) $\hat{c}(w)$ is strictly concave and either (a) there does not exist $\tilde{w}$ such that $\hat{c}(\tilde{w}) = 1$ or (b) $\hat{c}(\tilde{w})/\tilde{w} < 1$ for some $\tilde{w} \geq 0$, then there exists $\bar{w} > 0$ such that $\hat{c}(\bar{w}) = \bar{w}$.*

Thus under the assumptions of this lemma, there exists $\bar{w} > 0$ such that

$$c_t = c(w_t) \begin{cases} w_t & \text{if } w_t \leq \bar{w} \\ \hat{c}(w_t) & \text{if } w_t > \bar{w} \end{cases}. \tag{2.4}$$

[3] I assume the same K and ϕ satisfy the conditions below, but one can allow for different functions ϕ to satisfy each of them.

For simplicity, let $R_{t+1} = R$ for all $t = 0, 1, 2, \ldots$ and let $\mathcal{Y}$ denote the set of values which the income process can take.

Assumption B. The income process, $\{y_t\}$ is such that $y_t \in \mathcal{Y}$ is identically and independently distributed across periods in one of the following two ways:

(i) $\mathcal{Y} = \{y^1, y^2, \ldots, y^n\}$, where $y^1 \leq y^2 \leq \cdots \leq y^n$, and each y^i occurs with probability $\Gamma_i > 0$.

(ii) $\mathcal{Y} = [y^1, y^n) \subseteq \mathbb{R}_+$ with distribution Γ that is absolutely continuous with a lower semicontinuous density γ_y on $\mathcal{Y}$.

Denote by $\mathfrak{B}(\mathcal{Y})$ the Borel σ-algebra of $\mathcal{Y}$ and for any $\mathcal{A} \in \mathfrak{B}(\mathcal{Y})$ let $\Gamma(\mathcal{A})$ denote the probability of event $\mathcal{A}$ under (i) or (ii). We will use $\underline{y} = y^1$ and $\bar{y} = y^n$ whenever the use of a double superscript might be confusing.

Clearly, under assumptions A and B, the wealth process $\{w_t\}$ is Markov with state space $(\mathcal{W}, \mathfrak{B}(\mathcal{W})) = (\mathbb{R}_+, \mathfrak{B}(\mathbb{R}_+))$ and transition probability kernel P defined by

$$P(w, \mathcal{A}) = \Gamma\Big(\big\{\mathcal{A} - R\big(w - c(w)\big)\big\} \cap \mathcal{Y}\Big), \qquad \forall w \in \mathcal{W},\ \mathcal{A} \in \mathfrak{B}(\mathcal{W}). \quad (2.5)$$

$P(w, \mathcal{A})$ gives the probability of going from state w to a set $\mathcal{A}$ in one period. Notice that if $w \in [0, \bar{w}]$, then $P(w, \mathcal{A}) = \Gamma(\mathcal{A} \cap \mathcal{Y})$ independent of w, i.e. $[0, \bar{w}]$ is an atom (Meyn and Tweedie, 1993, p.103). If $y^1 \leq \bar{w}$, then as will be seen from the proof of Theorem 2.2 this set is an accessible atom. Define

$$\mathcal{A}_l = \big\{w \in \mathcal{W} \mid w = R\big(w - c(w)\big) + y^l\big\}$$

$$w^l = \begin{cases} \sup \mathcal{A}_l & \text{if } \mathcal{A}_l \neq \emptyset \\ \infty & \text{otherwise} \end{cases}, \qquad l = 1, n$$

$$^l w = \begin{cases} \inf \mathcal{A}_l & \text{if } \mathcal{A}_l \neq \emptyset \\ \infty & \text{otherwise} \end{cases}, \qquad l = 1, n.$$

Notice that $y^1 \leq {}^1w < {}^nw \leq w^n \leq \infty$, $w^1 < \infty$. If $w^n < \infty$ and ${}^1w < w^1$, then $w^n < w^1$.

Theorem 2.2. *If assumptions A and B hold, then:*

(i) If $w^n < \infty$, then there exists a unique invariant probability measure π on $\mathcal{W}$ and a π-null set N such that for any initial distribution of initial wealth λ,[4] *such that $\lambda(N) = 0$, and*

$$\left\| \int \lambda(dw) P^m(w, \cdot) - \pi(\cdot) \right\| \to 0, \quad m \to \infty.$$

(ii) If $w^n = \infty$, then $P(w_t \to \infty) = 1$.

The strength of this theorem is that, unlike most of the literature, it *does not require* wealth to be bounded for the existence of a unique ergodic distribution and it gives an easily verifiable condition. Clearly a necessary condition for the existence of a unique invariant distribution, under this theorem is:

Assumption C. $y^n < \infty$.

[4] Here $\|\cdot\|$ denotes the total variation norm, i.e. if λ is a signed measure on $\mathfrak{B}(\mathcal{W})$ then

$$\|\lambda\| := \sup_{f:|f|\leq 1} |\lambda(f)|.$$

Under this assumption, if $w^n < \infty$, then the stationary distribution π satisfies

$$\pi(A) = \begin{cases} > 0 & \text{if } A \cap [y^1, {}^n w] \neq \emptyset \\ = 0 & \text{if } A \cap [y^1, {}^n w] = \emptyset \end{cases}, \qquad \pi\left([y^1, {}^n w]\right) = 1. \tag{2.6}$$

3 Consumption rules

We will analyze the behavior of two families of consumption functions, one generated by the optimal solution to the problem

$$\max_{s.t.} \qquad \sum_{t=0}^{\infty} E_0\left[\beta^t u(c_t)\right] \tag{3.1a}$$

$$w_{t+1} = R_{t+1}(w_t - c_t) + y_{t+1} \tag{3.1b}$$

$$0 \leq c_t \leq w_t \tag{3.1c}$$

$$w_0 \geq 0 \text{ given,}$$

which I will denote by $c^*(w)$ and the family of linear consumption functions of the form

$$c^b(\alpha, w) = c^b(\alpha_0, \alpha_1, w) = \min\left\{\alpha_0 + \alpha_1 w, w\right\}, \qquad \alpha_0 \in [0, \bar{\alpha}_0],\ \alpha_1 \in [0, \bar{\alpha}_1], \tag{3.2}$$

where I will use $\alpha = (\alpha_0, \alpha_1)$ whenever this does not create confusion. Clearly, if $\alpha_1 \geq 1$, then $c^b(\alpha, w) = w$.

Given the recursive nature of (3.1), $c^*(w)$ can be analyzed as the solution to

$$v(w) = \max_{s.t.\ 0 \le c \le w} u(c) + \beta E\Big[v(h(c,w,y)) \mid c, w\Big] \tag{3.3a}$$

$$c^*(w) = \operatorname*{argmax}_{s.t.\ 0 \le c \le w} u(c) + \beta E\Big[v(h(c,w,y)) \mid c, w\Big] \tag{3.3b}$$

where $h(c,w,y) = R\big(w - c\big) + y$. I shall assume that under assumptions B and C, the optimal consumption function is continuous, concave and such that

$$c^*(w) = \begin{cases} w & \text{if } w \le \bar{w} \\ c(w) & \text{if } w > \bar{w} \end{cases} \tag{3.4}$$

for a unique $\bar{w} > y^1$ and a strictly concave function $c(w)$, which satisfies the first order conditions of (3.3a). Additionally, I assume that $w^n < \infty$ under the optimal rule, i.e. there exists a solution to the equation $w^n = R(w^n - c^*(w^n)) + y^n$. In particular, consider the following case:

Theorem 3.1. *If $R\beta < 1$ and*

$$u(c_t) = \begin{cases} \frac{c_t^{1-\theta} - 1}{1-\theta} & \textit{if } \theta \neq 1 \\ \ln c_t & \textit{if } \theta = 1 \end{cases},$$

then $c^*(w)$ satisfies all the required properties.

Clearly, the function $c^*(w)$ is independent of w_0, so for any rational agent, if she knows that w_0 is distributed according to some distribution λ, her ex-ante expected lifetime utility, before she learns the value of w_0, is given by

$$EV \equiv \int_W v(w)\lambda(dw). \tag{EV}$$

Define

$$CE \equiv u^{-1}(EV \cdot (1-\beta)) \tag{CE}$$

as the certainty equivalent of this utility. If the wealth process generated by $c^*(w)$ satisfies $w^n < \infty$, then from theorem 2.2 there exists a unique ergodic invariant distribution over wealth π^*. In this case, let EV^* and CE^* be the values implied by (EV) and (CE) when $\lambda = \pi^*$. I will use deviations from CE^* below in order to measure the performance of the different linear rules.

For a given w_0 and a fixed linear rule $c^b(\alpha, w)$ with parameters α_0, α_1 the expected life-time utility of an agent is

$$U(\alpha_0, \alpha_1, w_0) = \sum_{t=0}^{\infty} E_0\Big[\beta^t u\Big(\min\{\alpha_0 + \alpha_1 w_t, w_t\}\Big)\Big], \tag{3.5}$$

where w_t evolves according to (3.1b). Just as I did for $c^*(w)$, I can define the ex-ante expected lifetime utility and the certainty equivalent for a given rule as

$$EV_{\alpha_0,\alpha_1} \equiv \int_W U(\alpha_0, \alpha_1, w)\lambda(dw) \tag{EV^b}$$

$$CE_{\alpha_0,\alpha_1} \equiv u^{-1}(EV_{\alpha_0,\alpha_1} \cdot (1-\beta)) \tag{CE^b}$$

If $\lambda = \pi^*$ I will denote them as $EV^*_{\alpha_0,\alpha_1}$ and $CE^*_{\alpha_0,\alpha_1}$. If the linear consumption rule implies $w^n < \infty$, let π_{α_0,α_1} be the unique invariant distribution determined by the rule and denote by $EV^b_{\alpha_0,\alpha_1}$ and $CE^b_{\alpha_0,\alpha_1}$ the ex-ante expected lifetime utility and the certainty equivalent when $\lambda = \pi_{\alpha_0,\alpha_1}$ and the linear consumption rule is used, and EV^*_b and CE^*_b when the optimal consumption rule is used.

Under $c^b(\alpha, w)$ a necessary condition for $w^n < \infty$ is that either

$$\bar{w}_\alpha \equiv \frac{\alpha_0}{1-\alpha_1} \geq y^n \qquad \text{or} \qquad \alpha_1 \geq \frac{R-1}{R} \tag{3.6}$$

in which case, the process is positive Harris recurrent. Notice also that if $\frac{\alpha_0}{1-\alpha_1} \geq y^n$, then $y^n = {}^n w = w^n$ and the consumer is always constrained, which cannot be optimal. Thus, I shall assume $\alpha_1 < 1$.In order to be able to prove the existence of an optimal linear consumption function I will need the following assumptions.

Assumption D. Let γ_y be continuously differentiable, Lipschitz continuous and such that $\int y\gamma_y(y)dy < \infty$.

Assumption E. $\int_{\mathcal{Y}} \gamma_y(y)\phi(y)dy < \infty$ where $\phi(\cdot)$ is defined in (2.3).

Assumption F. The random variable $\xi_\alpha(w) = \phi(w')/\phi(w)$, where $w' = R(w - c^b(\alpha, w)) + y$, is uniformly integrable in w and α.

Notice that $CE^* \geq CE^*_\alpha$ for any α, while $CE^*_\alpha \geq CE^*_{\alpha'}$ if, and only if, $EV^*_\alpha - EV^* \geq EV^*_{\alpha'} - EV^*$, and similarly for the other stationary distributions.

Theorem 3.2. *(i) There exists* $(\alpha^*_0, \alpha^*_1) \in [0, \bar{\alpha}_0] \times [0, \bar{\alpha}_1]$ *which solves*

$$\max_{(\alpha_0,\alpha_1)\in[0,\bar{\alpha}_0]\times[0,\bar{\alpha}_1]} EV^*_{\alpha_0,\alpha_1} - EV^*. \tag{3.7}$$

(ii) If assumptions B.(ii), C, D and F hold and either (a) $y^1 > 0$, or (b) assumption E holds, then there exists $(\alpha_0^*, \alpha_1^*) \in [0, \bar{\alpha}_0] \times [(R-1)/R, \bar{\alpha}_1]$ which solves

$$\max_{(\alpha_0,\alpha_1)\in[0,\bar{\alpha}_0]\times[(R-1)/R,\bar{\alpha}_1]} EV^b_{\alpha_0,\alpha_1} - EV^*_b. \tag{3.8}$$

(iii) If assumptions B.(ii), C, D and F hold and either (a) $y^1 > 0$ or (b) assumption E holds, then there exists $(\alpha_0^*, \alpha_1^*) \in [0, \bar{\alpha}_0] \times [(R-1)/R, \bar{\alpha}_1]$ which solves

$$\max_{(\alpha_0,\alpha_1)\in[0,\bar{\alpha}_0]\times[(R-1)/R,\bar{\alpha}_1]} EV^b_{\alpha_0,\alpha_1} - EV^*. \tag{3.9}$$

The linear consumption rule implied by (α_0^*, α_1^*), $c^b(\alpha^*, w)$, is called the optimal linear consumption rule.

Howitt and Özak (2009) show that in their numerical exercise, there is little difference in these various measures. From a normative perspective it might make more sense to focus on problem (3.9), since it measures the willingness of agents to pay in order to adopt the optimal consumption rule $c^*(w)$ by comparing the utilities generated in the stationary distributions under bounded rationality and unbounded rationality. Analytically, most results for problem (3.9) can be generated from the analysis of problem (3.8) by adjusting certain steps in the proof, in particular, by setting $v(w_t) = 0$ whenever this term appears in the proofs. Additionally, notice that if α^* solves (3.8), then it also solves (3.9). So, I shall not present proofs for them and focus mostly on problem (3.8).

In order to be able to relate the solutions to the HO-learning algorithm, it is useful to have some characterizations of the optimal linear rule. For this, the following proposition might prove useful.

Proposition 3.3. *If* $\lim_{t\to\infty} \beta^t E_0\Big[U(\alpha_0,\alpha_1,w_t) - v(w_t)\Big] = 0$ *for all* (α_0,α_1) *then:*

(i)

$$U(\alpha_0,\alpha_1,w_t) - v(w_t) = \\ E_0\left[\sum_{j=0}^{\infty} \mu(w_{t+j})(c^b(w_{t+j}) - c^*(w_{t+j})) - k_{t+j}(c^b(w_{t+j}) - c^*(w_{t+j}))^2\right]. \tag{3.10}$$

(ii)

$$\begin{aligned} EV^*_{\alpha_0,\alpha_1} - EV^* = \\ &\int_{\mathcal{W}} E_0\left[\sum_{j=0}^{\infty} \beta^j \Big(\mu(w_{t+j})(c^b(w_{t+j}) - c^*(w_{t+j}))\right. \\ &\left.-k_{t+j}(c^b(w_{t+j}) - c^*(w_{t+j}))^2\Big)\right] \pi^*(dw_t). \end{aligned} \tag{3.11}$$

(iii)

$$\begin{aligned} EV^b_{\alpha_0,\alpha_1} - EV^*_b = \\ &\sum_{j=0}^{\infty}\beta^j \int_{\mathcal{W}} \big[\mu(w_{t+j})(c^b(w_{t+j}) - c^*(w_{t+j})) \\ &-k_{t+j}(c^b(w_{t+j}) - c^*(w_{t+j}))^2 \pi_{\alpha_0,\alpha_1}(dw_{t+j})\big]. \end{aligned} \tag{3.12}$$

Since the stationary distribution has positive mass only on $[y^1, {}^n w]$, this proposition implies that:

Corollary 3.4. *The solution* (α_0^*, α_1^*) *to* (3.8) *belongs to the set*

$$\Lambda = \left\{(\alpha_0, \alpha_1) \in [0, \bar{\alpha}_0] \times [(R-1)/R, \bar{\alpha}_1] \mid \alpha_0 + \alpha_1 \bar{w} \geq \bar{w}\right\},$$

and thus, solves the problem

$$\min_{(\alpha_0,\alpha_1)\in\Lambda} \sum_{j=0}^{\infty} \beta^j \int_{\mathcal{W}} \left[k_{t+j}(c^b(w_{t+j}) - c^*(w_{t+j}))^2 \pi_{\alpha_0,\alpha_1}(dw_{t+j})\right], \tag{3.13}$$

which is equivalent to

$$\min_{(\alpha_0,\alpha_1)\in\Lambda} \int_{\mathcal{W}} \left[k_s(c^b(w_s) - c^*(w_s))^2 \pi_{\alpha_0,\alpha_1}(dw_s)\right] \tag{3.14}$$

Thus, under our assumptions, the bounded rationality solution minimizes the expected squared difference from the optimal consumption function, which allows some additional and useful characterizations. In particular,

Theorem 3.5. *If* (α_0^*, α_1^*) *solves* (3.8)*, then it also solves the following problem*

$$\min_{(\alpha_0,\alpha_1)\in\Lambda} \int_{\mathcal{W}} \left(\beta R E_t u'(c^b(w_{t+1})) - u'(c^b(w_t))\right)^2 \pi_\alpha(dw_t), \tag{3.15}$$

If the solution is interior, then (α_0^*, α_1^*) *satisfies*

$$\begin{aligned} &\int_{\mathcal{W}} -2u''(c^b(w_t))\Big(\beta R E_t u'(c^b(w_{t+1})) - u'(c^b(w_t))\Big) \begin{pmatrix} 1 \\ w_t \end{pmatrix} \pi_\alpha(dw_t) \\ &\qquad + \int_{\mathcal{W}} \Big(\beta R E_t u'(c^b(w_{t+1})) - u'(c^b(w_t))\Big)^2 \begin{pmatrix} \frac{\partial \pi_\alpha(dw_t)}{\partial \alpha_0} \\ \frac{\partial \pi_\alpha(dw_t)}{\partial \alpha_1} \end{pmatrix} = 0. \end{aligned} \tag{3.16}$$

Rewrite this last condition as

$$f_0(\alpha^*) + g_0(\alpha^*) = 0, \qquad\qquad f_1(\alpha^*) + g_1(\alpha^*) = 0$$

where

$$\begin{aligned} f_0(\alpha) &= \int -2u''(c^b(w_t))\Big(\beta R E_t u'(c^b(w_{t+1})) - u'(c^b(w_t))\Big)\pi_\alpha(dw_t) \\ g_0(\alpha) &= \int \Big(\beta R E_t u'(c^b(w_{t+1})) - u'(c^b(w_t))\Big)^2 \frac{\partial \pi_\alpha}{\partial \alpha_0}(dw_t) \\ f_1(\alpha) &= \int -2u''(c^b(w_t))\Big(\beta R E_t u'(c^b(w_{t+1})) - u'(c^b(w_t))\Big) w \pi_\alpha(dw_t) \\ g_1(\alpha) &= \int \Big(\beta R E_t u'(c^b(w_{t+1})) - u'(c^b(w_t))\Big)^2 \frac{\partial \pi_\alpha}{\partial \alpha_1}(dw). \end{aligned}$$

Call $c^b(\alpha^q, w)$ a *quasi-optimal linear consumption rule*, if $\alpha^q = (\alpha_0^q, \alpha_1^q)$ satisfies

$$f_0(\alpha^q) = 0 \qquad \text{and} \qquad f_1(\alpha^q) = 0.$$

Since f_0 and f_1 are $\mathscr{C}^1$, if

$$\begin{vmatrix} \frac{\partial}{\partial\alpha_0} f_0(\alpha^*) & \frac{\partial}{\partial\alpha_1} f_0(\alpha^*) \\ \frac{\partial}{\partial\alpha_1} f_1(\alpha^*) & \frac{\partial}{\partial\alpha_1} f_1(\alpha^*) \end{vmatrix} \neq 0,$$

then by the Implicit Function Theorem, there exist $\mathscr{C}^1$ functions $\alpha_0(\zeta_0, \zeta_1)$ and $\alpha_1(\zeta_0, \zeta_1)$ such that

$$\alpha_0(g_0(\alpha^*), g_1(\alpha^*)) = \alpha_0^*, \qquad \alpha_1(g_0(\alpha^*), g_1(\alpha^*)) = \alpha_1^*,$$

$$f_0(\alpha(\zeta_0, \zeta_1)) + \zeta_0 = 0, \qquad f_1(\alpha(\zeta_0, \zeta_1)) + \zeta_1 = 0$$

for (ζ_0, ζ_1) in a certain open set around $\zeta^* = (g_0(\alpha^*), g_1(\alpha^*))$. Call this open set Λ_{ζ^*}. Thus, I have proven that

Proposition 3.6. *A quasi-optimal linear consumption function exists if* $(0, 0) \in \Lambda_{\zeta^*}$.

But this implies that

Corollary 3.7. *If*

$$\begin{vmatrix} \frac{\partial}{\partial\alpha_0} f_0(\alpha) & \frac{\partial}{\partial\alpha_1} f_0(\alpha) \\ \frac{\partial}{\partial\alpha_1} f_1(\alpha) & \frac{\partial}{\partial\alpha_1} f_1(\alpha) \end{vmatrix} \neq 0,$$

for all $\alpha \in [0, \bar{\alpha}_0] \times [(R-1)/R, \bar{\alpha}_1]$, *then there exists a quasi-optimal linear consumption function.*

Furthermore,

Corollary 3.8. *If the objective function in* (3.15) *is a strictly convex and* *twice continuously differentiable function of α, then there exists a unique quasi-optimal linear consumption function.*

In the setting of the last two corollaries, the functions $\alpha_0(\zeta_0, \zeta_1)$ and $\alpha_1(\zeta_0, \zeta_1)$ are open maps, and so, if $(g_0(\alpha^*), g_1(\alpha^*))$ is close to $(0, 0)$, then so is α^* from α^q. In particular, let $\bar{\Lambda}_\zeta = \cap_{n=0}^{\infty} A_n \subseteq \Lambda_{\zeta^*}$, where $\{A_n\}$ is a sequence of convex and closed subsets of Λ_{ζ^*} such that $(g_0(\alpha^*), g_1(\alpha^*)), (g_0(\alpha^q), g_1(\alpha^q)) \in A_n$ for all $n \geq 0$, and $A_n \subseteq A_{n-1}$. Then $\|\alpha^q - \alpha^*\| \leq \sup_{\alpha \in \bar{\Lambda}_{\zeta^*}} \|\alpha - \alpha^*\| \leq \sup_{\alpha, \alpha' \in \bar{\Lambda}_{\zeta^*}} \|\alpha - \alpha'\| = d(\bar{\Lambda}_{\zeta^*})$, i.e. the distance between the quasi-optimal and optimal consumption rules is smaller than the diameter of the set $\bar{\Lambda}_\zeta$.

Corollary 3.9. *If the objective function in* (3.15) *is a strictly convex and twice continuously differentiable function of α, then the difference in the objective function of* (3.8) *or* (3.9) *evaluated at $\alpha = \alpha^*$ and $\alpha - \alpha^q$ satisfies*

$$0 \leq (EV^b_{\alpha_0, \alpha_1} - EV^*_b)|_{\alpha^*} - (EV^b_{\alpha_0, \alpha_1} - EV^*_b)|_{\alpha^q} \leq k_1 d(\bar{\Lambda}_{\zeta^*}),$$

and

$$0 \leq EV^b_{\alpha^*} - EV^b_{\alpha^q} \leq k_2 d(\bar{\Lambda}_{\zeta^*}),$$

where k_1 and k_2 are upper bounds of the second derivatives of the objective functions.

Thus, the "flatter" the objective functions are, the closer is the utility under the quasi-optimal linear function to the optimal linear one. This result is similar to the one found by Akerlof and Yellen (1985) in their analysis of bounded rationality. Clearly, if $(g_0(\alpha^*), g_1(\alpha^*)) = (0, 0)$, then $\alpha^* = \alpha^q$. Although this requirement might not hold in general, the following result proves that it holds for certain classes of stationary distributions π_α.

In particular, Heidergott and Vázquez-Abad (2008) show that the measure-valued derivative of a distribution π_α can be written as a triplet $(k_\alpha, \pi_\alpha^-, \pi_\alpha^+)$, where $k_\alpha \in \mathbb{R}_{++}$, and $\pi_\alpha^-, \pi_\alpha^+$ are two probability measures. For example, if income is uniformly distributed between y^1 and y^n, then $\gamma(y) = \frac{1}{y^n - y^1}$, $\pi_\alpha(w) = \frac{w - y^1}{{}^n w - y^1}$, and $\frac{\partial \pi_\alpha}{\partial \alpha_i}$, $i = 0, 1$, can be written as

$$(k_\alpha^i, \pi_\alpha^-, \pi_\alpha^+) = \left(\frac{1}{{}^n w - y^1} \frac{\partial\, {}^n w}{\partial \alpha_i}, \delta({}^n w), \frac{w - y^1}{{}^n w - y^1} \right) \qquad i = 0, 1.$$

Theorem 3.10. *Assume that*

$$\frac{\partial \pi_\alpha}{\partial \alpha_0} = (k_\alpha^0, \pi_\alpha^-, \pi_\alpha^+), \qquad \text{and} \qquad \frac{\partial \pi_\alpha}{\partial \alpha_1} = (k_\alpha^1, \pi_\alpha^-, \pi_\alpha^+), \tag{3.17}$$

where $k^0 \neq k^1$. If α^* is an optimal linear consumption function, then it is a quasi-optimal linear consumption function.

If agents are boundedly rational, so that they are unable (or unwilling) to solve the maximization problem in (3.1) or in (3.8), it would seem that only by chance would agents behave optimally. In the next section I study the learning algorithm proposed by Howitt and Özak (2009) for boundedly rational agents and show conditions that allow agents to learn the quasi-optimal and optimal linear rules.

4 Learning the optimal consumption rule

The algorithm proposed by Howitt and Özak (2009) assumes that agents change their consumption rule based on how close marginal utility under that consumption rule is from next period's marginal utility of consumption, i.e. on $\beta u'(\min\{w_{t+1}, \alpha_t^0 + \alpha_t^1 w_{t+1}\}) - u'(\alpha_t^0 + \alpha_t^1 w_t)$ where w_{t+1} evolves according to (3.1b). Letting $c_{t+1} = \min\{w_{t+1}, \alpha_t^0 + \alpha_t^1 w_{t+1}\}$ denote *actual* consumption in period $t+1$ given the current rule, the algorithm assumes that agents change their consumption rule, given their current information, if they would have regretted using this rule given their actual consumption and last period's wealth, where their regret is measured by the difference $\beta u'(c_{t+1}) - u'(\alpha_t^0 + \alpha_t^1 w_t)$.

This method of learning uses very little information, especially when compared with methods that require an estimate of the value function (3.3a) as in Allen and Carroll (2001), or which compare many rules simultaneously as in Lettau and Uhlig (1999). The intuition behind the algorithm is the same that early marginalists used to explain how consumption across goods was determined. Agents would like to make the Euler equation hold with equality every period in order to assure they are behaving optimally, unless they are liquidity constrained. A nice quality of the algorithm is that agent's require very little information and keep track of only a few values, *independently* of the number of states or of the possible number of linear consumption rules.

Given some initial consumption rule with parameters (α_0^0, α_1^0), some initial wealth w_0 and some past consumption c_0, agents update their consumption rule using the following learning rule[5]

$$
\begin{aligned}
\begin{pmatrix} \alpha^0_{t+1} \\ \alpha^1_{t+1} \end{pmatrix} &= \begin{pmatrix} \alpha^0_t \\ \alpha^1_t \end{pmatrix} \\
&\quad + \kappa_t M_t^{-1} \left[\big(\beta R u'(c_{t+1}) - u'(\alpha^0_t + \alpha^1_t w_t) \big) u''(\alpha^0_t + \alpha^1_t w_t) \right] \begin{pmatrix} 1 \\ w_t \end{pmatrix}
\end{aligned} \tag{DG}
$$

$$
\begin{aligned}
M_{t+1} = & M_t + \kappa_t \Big\{ \Big[\big(u'(\alpha^0_t + \alpha^1_t w_t) - \beta R u'(c_{t+1}) \big) u'''(\alpha^0_t + \alpha^1_t w_t) \\
& + \Big(u''(\alpha^0_t + \alpha^1_t w_t) \Big)^2 \Big] \cdot \begin{pmatrix} 1 & w_t \\ w_t & w_t^2 \end{pmatrix} - M_t \Big\}
\end{aligned}
$$

for all $t \geq 0$. The algorithm can be seen as an approximation to the recursive representation of the solution of the following non-linear least squares problem[6]

$$
\min_{s.t.} \qquad \frac{1}{t-1} \sum_{k=0}^{t-2} \big(u'(\alpha^0_t + \alpha^1_t w_k) - \beta R u'(c_{k+1}) \big)^2. \tag{NLS}
$$

The literature on stochastic approximations to recursive algorithms (Benveniste, Métivier, and Priouret, 1990; Kushner and Yin, 2003) studies the dynamics of recursive algorithms like (ODE-DG) by using a differential equation obtained by averaging the dynamics of the algorithm as time evolves. In this

[5] I will present the same version of the algorithm used by Howitt and Özak (2009) since my results can be extended to cases when the matrix does evolves differently, and the same proofs can be used if it stays constant. This is particularly important, since their assumption on the evolution of M_t seems to be the less believable for behavior under bounded rationality. The main reason for using this version is that it seems to generate the fastest rates of convergence in numerical simulations.

[6] Ljung and Söderström (1983) provide a general analysis of this type of algorithms and how one can relate the HO-algorithm to the solution of this minimization problem.

case, the following ordinary differential equation related to the HO-algorithm (DG) is of interest:

$$
\begin{aligned}
\begin{pmatrix} \frac{d\alpha_0}{\partial\tau} \\ \\ \frac{d\alpha_1}{\partial\tau} \end{pmatrix} &= \int_{\mathcal{W}} M^{-1}\left[\big(\beta R E_t u'(c(w')) - u'(\alpha_0+\alpha_1 w)\big)u''(\alpha_0+\alpha_1 w)\right]\cdot \\
&\qquad \cdot \begin{pmatrix} 1 \\ \\ w \end{pmatrix} \pi_{\alpha_0,\alpha_1}(dw) \qquad\qquad \text{(ODE-DG)} \\
\frac{dM}{d\tau} &= \int_{\mathcal{W}} \Big(\Big[\big(u'(\alpha_0+\alpha_1 w) - \beta R E_t u'(c(w'))\big)u'''(\alpha_0+\alpha_1 w) \\
&\qquad + \Big(u''(\alpha_0+\alpha_1 w)\Big)^2\Big]\cdot \begin{pmatrix} 1 & w \\ \\ w & w^2 \end{pmatrix} - M\Big)\,\pi_{\alpha_0,\alpha_1}(dw)
\end{aligned}
$$

where E_t denotes the expectation using Γ, $c(w') = \min\{w', \alpha_0 + \alpha_1 w'\}$ and $w' = R(w - c(w)) + y$. I assume that $(\alpha(t), M(t))$ belongs to an open set $\mathcal{Q} \subset [0, \bar{\alpha}_0] \times [(R-1)/R, \bar{\alpha}_1] \times \mathbb{R}^4$. Let $h(\alpha, M) = (h_1(\alpha, M), h_2(\alpha, M))^T$ denote the right hand side of (ODE-DG), $(\alpha_0^e, \alpha_1^e, M^e)$ denote an equilibrium of (ODE-DG), and let $\nabla h_1(\alpha^e, M^e)$ denote the derivative of the first equation in (ODE-DG) with respect to (α_0, α_1) evaluated at the equilibrium.

Assumption G. $(\alpha_0^e, \alpha_1^e, M^e)$ is locally unique and $\nabla h_1(\alpha^e, M^e)$ is symmetric and definite negative.

Proposition 4.1. *If the assumptions of theorem 3.2.(ii) and assumption G hold, then $(\alpha_0^e, \alpha_1^e, M^e)$ is locally asymptotically stable.*

Let $\mathcal{Q}^e = \mathcal{Q}_\alpha^e \times \mathcal{Q}_M^e$ be the domain of attraction of $(\alpha_0^e, \alpha_1^e, M^e)$.

Corollary 4.2 (Krasovskiĭ (1963)). *Under the assumptions of proposition 4.1, there exists a function L on $\mathcal{Q}^e$ of class $\mathscr{C}^2$ such that*

(i) $L(\alpha^e, M^e) = 0$, $L(\alpha, M) > 0$ *for all* $(\alpha, M) \in \mathcal{Q}^e$, $(\alpha, M) \neq (\alpha^e, m^e)$.

(ii) $\nabla L(\alpha, M) \cdot h(\alpha, M) < 0$ *for all* $(\alpha, M) \in \mathcal{Q}^e$, $(\alpha, M) \neq (\alpha^e, m^e)$.

(iii) $L(\alpha, M) \to \infty$ *if* $(\alpha, M) \to \partial \mathcal{Q}^e$ *or* $\|(\alpha, M)\| \to \infty$.

The following assumptions are required in order to analyze the convergence of the algorithm to (α^e, M^e).

Assumption H. $\{\kappa_t\}_t$ is a decreasing sequence of positive real numbers, such that $\sum_t \kappa_t = \infty$ and $\sum_t \kappa_t^2 < \infty$.

Assumption I. $y^1 > 0$ and $\int y^2 \Gamma(dy) < \infty$.

For any $b \in \mathbb{R}_+$, let $K(b) = \{(\alpha, M) \in \mathcal{Q}^e \mid L(\alpha, M) \leq b\}$, and $\tau(b) = \inf \{n \in \mathbb{N} \mid (\alpha_n, M_n) \notin K(b)\}$. Also let $\mathcal{Q}_1 \subset \mathcal{Q}$ and $\mathcal{Q}_2 \subseteq \mathcal{Q}^e$ be compact sets and

$$\Omega(\mathcal{Q}_1, \mathcal{Q}_2) = \{(\alpha_n, M_n) \in \mathcal{Q}_1 \text{ for all } n, (\alpha_n, M_n) \in \mathcal{Q}_2 \text{ for infinitely many } n\}.$$

Theorem 4.3. *Let $b < b_1 < b_2 < \infty$. If the assumptions of proposition 4.1 and assumptions H and I hold, then:*

(i) There exist constants B_3 and s such that for all $(\alpha_0, M_0) \in K(b_1)$ and all $w \in \mathcal{W}$,

$$P_{w,(\alpha_0,M_0)}(\{\tau(b_2) < \infty\}) \leq B_3(1 + |w|^s) \sum_{k=1}^{\infty} \kappa_k^2. \tag{4.1}$$

(ii) *For all* $(\alpha_0, M_0) \in K(b)$ *and all* $w \in \mathcal{W}$, $(\alpha_n, M_n) \to (\alpha^e, M^e)$ $P_{w,(\alpha_0,M_0)}$-*a.s. on* $\{\tau(b_2) = \infty\}$.

(iii) *There exist constants* B_4 *and* s *such that for all* $n \geq 0$, *all* $(\alpha_0, M_0) \in \mathcal{Q}_2$ *and all* $w \in \mathcal{W}$

$$P_{n,w,(\alpha_0,M_0)}(\{(\alpha_n, M_n) \to (\alpha^e, M^e)\}) \geq 1 - B_4(1 + |w|^s) \sum_{k=n+1}^{\infty} \kappa_k^2. \quad (4.2)$$

(iv) *For all* $w \in \mathcal{W}$, $(\alpha_0, M_0) \in \mathcal{Q}_2$, $(\alpha_n, M_n) \to (\alpha^e, M^e)$ $P_{w,(\alpha_0,M_0)}$-*a.s. on* $\Omega(\mathcal{Q}_1, \mathcal{Q}_2)$.

$P_{w,(\alpha_0,M_0)}$ *denotes the distribution of* $\{(w_k, (\alpha_k, M_k))\}_{k\geq 0}$ *starting from* $(w, (\alpha_0, M_0))$ *and* $P_{n,w,(\alpha_0,M_0)}$ *denotes the distribution of* $\{(w_{n+k}, (\alpha_{n+k}, M_{n+k}))\}_{k\geq 0}$ *starting from* $(w, (\alpha_0, M_0))$.

This theorem gives bounds on the probability of convergence of the algorithm to the equilibrium of (ODE-DG) and of escaping its domain of attraction. It additionally ensures the convergence of the algorithm to the equilibrium when the initial conditions are in some subset of the domain of attraction. If a projection facility is used, such that if $(\alpha_t, M_t) \notin \mathcal{Q}^e$, then it projects the parameters into $\mathcal{Q}^e$, then a much stronger result follows:

Corollary 4.4. (α_t, M_t) *converges to* (α^e, M^e) *a.s.*

Notice that $h_1(\alpha^e, M) = 0$ for any matrix M. Thus, the matrix M plays only a role on the convergence of the system to α^e, but not on its value. This is particularly important, since one of the least appealing aspects of the H-O algorithm as a norm of behavior under bounded rationality is the way in which M_t is updated. Clearly, one can replace M_t and its updating process for a simpler process without affecting the value of α^e nor the convergence of the algorithm to it, as long as one maintains the stability properties shown in the previous results. In particular, letting $M_t = \bar{M}$ for all $t \geq 0$ such that $\nabla h_1(\alpha^e, \bar{M})$ is symmetric and definite negative, all the previous results hold without change.[7,8]

Clearly, any asymptotic equilibrium of (ODE-DG) is of interest for the analysis only in so far as it bears some relation with (α_0^*, α_1^*) or (α_0^q, α_1^q). The following theorem summarizes the main result:

Theorem 4.5. $\alpha = (\alpha_0, \alpha_1)$ *is a quasi-optimal linear consumption rule if, and only if, it is an equilibrium of* (ODE-DG). *If, additionally, the conditions of theorem 3.10 are satisfied, then the optimal linear consumption rule is an asymptotically stable equilibrium of* (ODE-DG).

So, under the assumptions of this theorem, agents employing the H-O algorithm *do learn to behave as if they were optimizing.*

5 Conclusions

The assumption of complete and perfect rationality has increasingly been criticized due, in part, to the high complexity of many solutions in economic models under this assumption. In response, models of bounded rationality and learning have recently flourished in economics, though the study and application

[7]This does not mean that the rate of convergence to the equilibrium is unchanged. Numerical simulations realized by the author suggest than under the original H-O algorithm convergence is much faster than under fixed arbitrary matrices.

[8]For example, let $\bar{M} = k \cdot I_{2\times 2}$, where $k \in \mathbb{R}_+$ and $I_{2\times 2}$ is the 2×2 identity matrix.

of these ideas to approximate solutions of stochastic dynamic programming problems is still an emerging area. In particular, the study of consumption-saving decisions under uncertainty and liquidity constraints has been pursued by only a couple of papers with limited or negative results.

In this paper I have shown that boundedly rational agents, who use a linear consumption function, are liquidity constrained, and have uncertain income, can learn to behave "optimally" by following the learning procedure proposed by Howitt and Özak (2009). In particular, I have provided conditions for the existence of an optimal linear consumption function and for the learning algorithm to converge to this optimum for a wide class of utility functions and income processes.

This is a first step towards studying the possibility of learning the solution to more complex dynamic programming problems. The problem at hand was simplified by the almost piecewise linear nature of the optimal consumption function, which allowed the use of a simple piecewise linear function as a norm for behavior under bounded rationality. It is my conjecture that it is possible to apply the techniques of this paper to learn the optimal solutions to more complex problems by using more general piecewise linear functions.

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Appendix

A Proofs

Proof of Lemma 2.1. (i) This follows directly from the intermediate value theorem.

(ii) We need to analyze two cases:

(a) If there does not exist $\tilde{w}$ such that $c(\tilde{w}) = 1$, then it must be that $0 < c(w) < 1$ for all $w > 0$ and so $\lim_{w\to\infty} c(w) - w = -\infty$ which from our previous result in (i) proves what I wanted.

(b) Since $c(w)$ is strictly increasing and strictly concave, I have that for any increasing sequence $\{w_n\}_{n=1}^{\infty}$ such that $w_n \to \infty$, for any $k \geq 1$ and $l > k$,

$$+\infty \geq \frac{c(w_l) - c(w_k)}{w_l - w_k} > \frac{c(w_{l+1}) - c(w_k)}{w_{l+1} - w_k} > \cdots > \lim_{n\to\infty} \frac{c(w_n) - c(w_k)}{w_n - w_k} \geq 0.$$

Clearly, the sequence $\left\{\frac{c(w_l)-c(w_k)}{w_l-w_k}\right\}$ is decreasing and bounded below, so that it converges to $\tau_k = \inf_n \frac{c(w_n)-c(w_k)}{w_n-w_k}$. On the other hand, I have that for any $w < w' < w''$

$$\frac{c(w') - c(w)}{w' - w} > \frac{c(w'') - c(w')}{w'' - w'}$$

so that the sequence $\{\tau_k\}_{k=1}^{\infty}$ is also decreasing and bounded below, thus the sequence converges to $\tau^* = \inf_k \tau_k$.

Notice that if $\tau^* \geq 1$, then

$$1 > \frac{c(\tilde{w})}{\tilde{w}} = \frac{c(\tilde{w}) - c(0)}{\tilde{w} - 0} \geq \tau_0 \geq \tau^* \geq 1$$

which is clearly a contradiction.

Thus, $\tau^* < 1$.

Fix $w^* > \hat{w}$, and define for $w > 0$ the function

$$f(w) = \frac{c(w^*) - c(\hat{w})}{w^* - \hat{w}}(w - \hat{w}) + c(\hat{w})$$

which is the linear function that goes through $(\hat{w}, c(\hat{w}))$ and $(w^*, c(w^*))$. We have that $\lim_{w\to\infty} f(w) - w = -\infty$ and also that $f(w) \geq c(w)$ for all $w > w^*$ by the strict concavity of $c(\cdot)$. Thus, again $\lim_{w\to\infty} c(w) - w = -\infty$. ■

Proof of Theorem 2.2. Here I present and prove a more general version of the theorem. For that, let $\{w_t\}$ be a Markov process with state space $(\mathcal{W}, \mathfrak{B}(\mathcal{W})) = (\mathbb{R}_+, \mathfrak{B}(\mathbb{R}_+))$, defined as

$$w_{t+1} = h(w_t) + y_{t+1}$$

for some increasing and convex (or concave) function $h(w)$ and transition probability kernel P defined by

$$P(w, \mathcal{A}) = \Gamma\Big(\big\{\mathcal{A} - h(w)\big)\big\} \cap \mathcal{Y}\Big), \qquad \forall w \in \mathcal{W},\ \mathcal{A} \in \mathfrak{B}(\mathcal{W}). \tag{A.1}$$

$P(w, \mathcal{A})$ gives the probability of going from state w to a set $\mathcal{A}$ in one period. Define

$$\mathcal{A}_l = \big\{w \in \mathcal{W} \mid w = h(w) + y^l\big\}$$

Theorem A.1. *If assumption B holds, then:*

(i) If $w^n < \infty$, then there exists a unique invariant probability measure π on $\mathcal{W}$ and a π-null set N such that for any initial distribution of initial wealth λ,[9] *such that $\lambda(N) = 0$,*

$$\left\| \int \lambda(dw) P^m(w, \cdot) - \pi(\cdot) \right\| \to 0, \quad m \to \infty.$$

(ii) If $w^n = \infty$, then $P(w_t \to \infty) = 1$.

Proof. We present the proof for the case when $h(w)$ is convex. For the other case, just revert the roles of w^1 and w^n.

(i) Assume that $w^n < \infty$. We need to analyze two cases:

[9] Here $\|\cdot\|$ denotes the total variation norm, i.e. if λ is a signed measure on $\mathfrak{B}(\mathcal{W})$ then

$$\|\lambda\| := \sup_{f:|f|\leq 1} |\lambda(f)|.$$

(a) If ${}^1w < w^1$, let $\mathcal{A}^* = [{}^1w, {}^n w]$, $\mathcal{A}^0 = [0, {}^1 w)$, $\mathcal{A}^1 = ({}^n w, w^1)$ and $\mathcal{A}^\infty = [w^1, \infty)$. Then

$$P(w, \mathcal{A}^*) \begin{cases} = 1 & \text{if } w \in \mathcal{A}^* \cup \mathcal{A}^0 \\ = 0 & \text{if } w \in \mathcal{A}^\infty \end{cases}, \qquad P(w, \mathcal{A}^\infty) \begin{cases} = 1 & \text{if } w \in \mathcal{A}^\infty \\ = 0 & \text{if } w \in \mathcal{A}^0 \cup \mathcal{A}^* \end{cases} \tag{A.2a}$$

$$P^m(w, \mathcal{A}^*) > 0 \;\text{ if } w \in \mathcal{A}^1 \qquad \text{or} \qquad P^{m'}(w, \mathcal{A}^\infty) > 0 \;\text{ if } w \in \mathcal{A}^1 \tag{A.2b}$$

for some $1 \leq m, m' < \infty$. So, for any $w \in \mathcal{W}$ and $\mathcal{A}$ such that $\mathcal{A}^* \subseteq \mathcal{A}$ and $\mathcal{A} \cap \mathcal{A}^\infty = [\hat{w}, \infty)$ for some $\hat{w} > w^1$, there exists $m < \infty$ such that $P^m(w, \mathcal{A}) > 0$. Letting

$$\varphi(\mathcal{A}) = \begin{cases} \Gamma(\mathcal{A} \cap \mathcal{Y}) & \text{if } \mathcal{A}^* \subseteq \mathcal{A} \text{ and } \mathcal{A} \cap \mathcal{A}^\infty = [\hat{w}, \infty) \text{ for some } \hat{w} \geq w^1 \\ 0 & \text{Otherwise} \end{cases}$$

I have that the process $\{w_t\}$ is φ-irreducible and thus there exists a maximal irreducibility measure ψ on $\mathfrak{B}(\mathcal{W})$ (Meyn and Tweedie, 1993, theorem 4.0.1). Furthermore, $\psi(\mathcal{A}^* \cup \mathcal{A}^\infty) > 0$, so that ψ has support with non-empty interior, and since the process is Feller, it is a T-chain by proposition 7.1.2 and theorem 6.0.1(iii) in Meyn and Tweedie (1993). Furthermore, theorem 6.0.1.(ii) ensures that every compact set is petite, and since $\mathcal{A}^*$ is compact and absorbent, theorem 8.3.6(i) ensures the process $\{w_t\}$ is recurrent. Thus, by theorem 10.4.4 I have the existence of a unique invariant measure $\bar{\pi}$, which, since $\mathcal{A}^*$ is petite and absorbing, by theorem 10.4.10 is finite, and equivalent to a probability measure π, so that the process is positive recurrent. Now, it is not hard to prove that the process is aperiodic, so that theorem 13.3.4(ii) in Meyn and Tweedie (1993) gives the desired result.

(b) If ${}^1w = w^1$, then also ${}^n w = w^n$. Let $\mathcal{A}^* = [w^1, w^n]$, $\mathcal{A}^0 = [0, w^1)$ and $\mathcal{A}^\infty = [w^n, \infty)$. Then

$$P(w, \mathcal{A}^*) = 1 \qquad \text{if } w \in \mathcal{A}^* \cup \mathcal{A}^0 \qquad \text{and} \qquad P^m(w, \mathcal{A}^*) \quad > 0 \qquad \text{if } w \in \mathcal{A}^\infty \tag{A.3}$$

for some $1 \leq m < \infty$. So, for any $w \in \mathcal{W}$ and $\mathcal{A}$ such that $\mathcal{A}^* \subseteq \mathcal{A}$, there exists

$m < \infty$ such that $P^m(w, \mathcal{A}) > 0$. Letting

$$\varphi(\mathcal{A}) = \begin{cases} \Gamma\left(\mathcal{A} \cap \mathcal{Y}\right) & \text{if } \mathcal{A}^* \subseteq \mathcal{A} \\ 0 & \text{Otherwise} \end{cases}$$

I have that the process $\{w_t\}$ is φ-irreducible and thus there exists a maximal irreducibility measure ψ on $\mathfrak{B}(\mathcal{W})$ (Meyn and Tweedie, 1993, theorem 4.0.1). Furthermore, since $\psi(\mathcal{A}^*) > 0$, the result follows from the same arguments as in (a).

(ii) If $w^n = \infty$, for any $w \in \mathcal{W}$ there exists $m < \infty$ such that $\epsilon_w = P^m(w, \mathcal{A}^\infty) > 0$, where $\mathcal{A}^\infty = [w^1, \infty)$. Define $\epsilon = \sup_{w \in [0,w^1)} \epsilon_w$. Clearly, $P(w, \mathcal{A}^\infty) = 1$ if $w \in \mathcal{A}^\infty$ and $P(w_t \to \infty \mid w_0 \in \mathcal{A}^\infty) = 1$, then for any $w \in [0, w^1)$,

$$P(w_t < \infty, \forall\, t) = 1 - P(w_t \to \infty) = P(w_t \in (\mathcal{A}^\infty)^C, \forall\, t) = \lim_{t\to\infty}(1-\epsilon_{w_t})^t \leq \lim_{t\to\infty}(1-\epsilon)^t = 0$$

so that $P(w_t \to \infty) = 1$. ■

It is easy to see that if $h(w)$ is concave, then the set N in the previous theorem is given by $N = \emptyset$. This theorem can easily be extended for other types of increasing functions, but this would require us to change the notation and analyze more subcases, so I do not pursue it here. ■

Proof of theorem 3.1. For this proof I follow Carroll (2004) closely.

The problem is not bounded, so the methods in Stokey, Lucas, and Prescott (1989) are not applicable in order to prove the existence and uniqueness of a measurable function v that satisfies equation (3.3a). For this, I need the contraction mapping theorem of Boyd III (1990).

Let $\phi : \mathbb{R}_+ \to \mathbb{R}_{++}$ be a continuous strictly positive function, define the ϕ-norm of a function f as $\|f\|_\phi = \sup \frac{|f(x)|}{\phi(x)}$ and define

$$C_\phi(\mathbb{R}_+, \mathbb{R}) = \left\{ f \in C(\mathbb{R}_+, \mathbb{R}) \mid \|f\|_\phi < K, \text{some } K > 0 \right\}$$

where $C(\mathbb{R}_+, \mathbb{R})$ is the set of continuous functions from $\mathbb{R}_+$ to $\mathbb{R}$,to be the set of ϕ-bounded continuous functions. Boyd III (1990) proofs the following:

Theorem A.2 (Weighted Contraction Mapping)**.** *Let $T : C_\phi(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$ be a map such that*

(i.) T is non-decreasing, i.e. $v \leq v'$ implies $T(v) \leq T(v')$.

(ii.) $T(0) \in C_\phi(\mathbb{R}_+, \mathbb{R})$.

(iii.) $T(v + a\phi) \leq T(v) + a\xi\phi$ for some $\xi < 1$ and all $a > 0$.

Then T has a unique fixed point v^*.

A corollary of this theorem is that starting from any $v_0 \in C_\phi(\mathbb{R}_+, \mathbb{R})$ I have that $lim_{n\to\infty} T^n(v_0) = v^*$.

Let

$$T(v(w)) = \max_{s.t.\ 0\leq c\leq w} u(c) + \beta E\Big[v(h(c, w, y)) \mid c, w\Big]$$

be the T-map defined by (3.3a), let's show that this T-map satisfies the conditions of theorem A.2. For this, define

$$\phi = \begin{cases} \eta + w + w^{1-\theta} & \text{if } \theta > 1 \\ \eta + w + w^{-1} & \text{if } \theta = 1 \\ \eta + w + w^{-\theta} + w^{1-\theta} & \text{if } \theta < 1 \end{cases}$$

where

$$\eta = \begin{cases} \frac{\beta(y_{max} + y_{min}^{1-\theta})}{\xi - \beta} & \text{if } \theta > 1 \\ \frac{\beta(y_{max} + y_{min}^{-1})}{\xi - \beta} & \text{if } \theta = 1 \\ \frac{\beta(y_{max} + y_{min}^{-\theta} + y_{max}^{1-\theta})}{\xi - \beta} & \text{if } \theta < 1 \end{cases}$$

for some $\xi \in (\beta, 1)$ such that $\beta R \leq \xi$.

(i.) If $v \leq v'$, then

$$\begin{aligned} T(v(w)) &= \max_{s.t.\ 0\leq c\leq w} u(c) + \beta E\Big[v(h(c, w, y)) \mid c, w\Big] \\ &\leq \max_{s.t.\ 0\leq c\leq w} u(c) + \beta E\Big[v'(h(c, w, y)) \mid c, w\Big] = T(v'(w)). \end{aligned}$$

(ii.) $T(0(w)) = \max_{s.t.\ 0\leq c\leq w} u(c) + \beta E\Big[0 \mid c, w\Big] = u(w)$. Clearly, $\|T(0)\|_\phi < K$ for some $K > 0$.

(iii.) Let $a > 0$, $y_{min} = y^1$, $y_{max} = y^n$, then if $\theta > 1$

$$\begin{aligned}
T(v + a\phi)(w) &= \max_{s.t.\ 0\leq c\leq w} u(c) + \beta E\Big[v(h(c,w,y)) + a\phi(h(c,w,y)) \mid c, w\Big] \\
&= \max_{s.t.\ 0\leq c\leq w} u(c) + \beta E\Big[v(h(c,w,y)) \mid c, w\Big] + \beta E\Big[a\phi(h(c,w,y)) \mid c, w\Big] \\
&\leq T(v(w)) + \max_{s.t.\ 0\leq c\leq w} a\beta E\Big[\eta + Rw - Rc + y + (Rw - Rc + y)^{1-\theta} \mid c, w\Big] \\
&\leq T(v(w)) + \max_{s.t.\ 0\leq c\leq w} a\left(\beta\eta + \beta Rw + \beta y_{max} + \beta E\Big[(Rw - Rc + y)^{1-\theta} \mid c, w\Big]\right) \\
&\leq T(v(w)) + a\left(\beta\eta + \beta Rw + \beta y_{max} + \beta y_{min}^{1-\theta} + \beta Rw^{1-\theta}\right) \\
&\leq T(v(w)) + a\xi\phi.
\end{aligned}$$

If $\theta = 1$

$$\begin{aligned}
T(v + a\phi)(w) &= \max_{s.t.\ 0\leq c\leq w} u(c) + \beta E\Big[v(h(c,w,y)) + a\phi(h(c,w,y)) \mid c, w\Big] \\
&= \max_{s.t.\ 0\leq c\leq w} u(c) + \beta E\Big[v(h(c,w,y)) \mid c, w\Big] + \beta E\Big[a\phi(h(c,w,y)) \mid c, w\Big] \\
&\leq T(v(w)) + \max_{s.t.\ 0\leq c\leq w} a\beta E\Big[\eta + Rw - Rc + y + (Rw - Rc + y)^{-1} \mid c, w\Big] \\
&\leq T(v(w)) + \max_{s.t.\ 0\leq c\leq w} a\left(\beta\eta + \beta Rw + \beta y_{max} + \beta E\Big[(Rw - Rc + y)^{-1} \mid c, w\Big]\right) \\
&\leq T(v(w)) + a\left(\beta\eta + \beta Rw + \beta y_{max} + \beta y_{min}^{-1} + \beta Rw^{-1}\right) \\
&\leq T(v(w)) + a\xi\phi.
\end{aligned}$$

If $0 < \theta < 1$

$$\begin{aligned}
T(v + a\phi)(w) &= \max_{s.t.\ 0\leq c\leq w} u(c) + \beta E\Big[v(h(c,w,y)) + a\phi(h(c,w,y)) \mid c, w\Big] \\
&= \max_{s.t.\ 0\leq c\leq w} u(c) + \beta E\Big[v(h(c,w,y)) \mid c, w\Big] + \beta E\Big[a\phi(h(c,w,y)) \mid c, w\Big] \\
&\leq T(v(w)) + \max_{s.t.\ 0\leq c\leq w} a\beta E\Big[\eta + Rw - Rc + y + (Rw - Rc + y)^{-\theta} + (Rw - Rc + y)^{1-\theta} \mid c, w\Big] \\
&\leq T(v(w)) + \max_{s.t.\ 0\leq c\leq w} a\left(\beta\eta + \beta Rw + \beta y_{max} + \beta y_{min}^{-\theta} + \beta E\Big[(Rw - Rc + y)^{1-\theta} \mid c, w\Big]\right) \\
&\leq T(v(w)) + a\left(\beta\eta + \beta Rw + \beta y_{max} + \beta y_{min}^{-\theta} + y_{max}^{1-\theta} + \beta Rw^{-\theta} + \beta Rw^{1-\theta}\right)
\end{aligned}$$

$$\leq T(v(w)) + a\xi\phi.$$

Thus, there exists a unique solution v^* to the functional equation defined by (3.3a) which is continuous. Since $u(\cdot)$ is differentiable, strictly increasing and strictly concave, I have that $T(v)$ is also strictly increasing and strictly concave and corollary 1 of Stokey, Lucas, and Prescott (1989, p.52) implies that v^* is also concave, and the theorem by Benveniste and Scheinkman (1979) implies that it is differentiable and strictly increasing. By the theorem of the maximum (see e.g. Stokey, Lucas, and Prescott, 1989, p.62) $c^*(w)$ exists and is continuous. These last two results imply that the consumption function is strictly increasing. Since for any $v_0 \in C_\phi(\mathbb{R}_+, \mathbb{R})$ the sequence $v_n = T^n(v_0)$ converges to v^*, the fact that $u(c)$ belongs to $C_\phi(\mathbb{R}_+, \mathbb{R})$ and also to the HARA family of utility functions implies that $c^*(w)$ is concave by the results of Carroll and Kimball (1996).

Since both v^* and u are concave and the constraint is convex, the unique solution $c^*(w)$ solves the related Kuhn-Tucker conditions (Arrow and Enthoven, 1961; Arrow, Hurwicz, and Uzawa, 1961). This implies that there exists $\bar{w} > 0$ as defined in the theorem. To see this, assume on the contrary that the agent is constrained for all wealth levels. This can happen if and only if there does not exist a solution to the first order condition

$$u'(\bar{w}) - \beta R E[u'(y)] = 0.$$

Let $f(w) = u'(w) - \beta R E[u'(y)]$, then $f(0) = +\infty$ and $\lim_{w\to\infty} f(w) < 0$. By the intermediate value theorem there exists a solution, contradicting our starting hypothesis, so that I conclude the existence of $\bar{w}$.

Finally, since at $\bar{w}$ the optimal consumption function satisfies

$$u'(\bar{w}) = \beta R E[u'(y)] < \beta R u'(y_{min}) < u'(y_{min}),$$

I have that $\bar{w} > y_{min}$. ■

Proof of Theorem 3.2. (i) Clearly, $U(\alpha_0, \alpha_1, w)$, $EV^*_{\alpha_0,\alpha_1}$ and EV^*_b are continuous, so the result follows by Weierstrass' Theorem.

(ii) Our proof of this result uses the theory of measure valued differentiation (Heidergott, Hordijk, and Weisshaupt, 2006; Heidergott and Vázquez-Abad, 2006, 2008). In order to prove this part, I need to show that under our assumptions, the proof of Theorem 4.2 of Heidergott, Hordijk, and Weisshaupt (2006) holds, so that π_{α_0,α_1} is $\mathscr{D}_\phi$-Lipschitz continuous and differentiable. For simplicity I will follow the notation and definitions used by Heidergott, Hordijk, and Weisshaupt (2006). We present the proof in steps.

Step 1: Let $\mathcal{A}^*$ be as in (i)(a) in the proof of Theorem 2.2. This set is petite and the first return time to $\mathcal{A}^*$ for every $w \in \mathcal{A}^*$, $\tau_{\mathcal{A}^*}$ is 1. So, theorem 14.0.1 of Meyn and Tweedie (1993) implies that for any function $f : \mathcal{W} \to [1, \infty)$ I have that

$$\int_{\mathcal{W}} f(w)\pi_{\alpha_0,\alpha_1}(dw) < \infty.$$

Clearly, for any continuous function $g : \mathcal{W} \to \mathbb{R}$, $|g| + 1$ satisfies the conditions above, so that $|g| + 1$ is π_{α_0,α_1}-integrable. By theorem 11.27 of Rudin (1966) g is also π_{α_0,α_1}-integrable.

Step 2: For fixed α, let $P(w, \mathcal{A}; \alpha)$ denote the one-step transition kernel. By (3.1b) $w' \in \mathcal{W}(\alpha, w) \equiv [R(w - c^b(\alpha, w)) + y^1, R(w - c^b(\alpha, w)) + y^n)$ with probability one, so that the density function of w' is

$$\gamma_{w'}(w'; \alpha, w) = \begin{cases} \gamma_y(w' - R(w - c^b(\alpha, w))) & \text{if } w' \in \mathcal{W}(\alpha, w) \\ 0 & \text{Otherwise} \end{cases}$$

and

$$P(w, \mathcal{A}; \alpha) = \int_{\mathcal{A}} \gamma_{w'}(w'; \alpha, w)dw'.$$

Under (a), this implies that for any continuous function $g : \mathcal{W} \to \mathbb{R}$,

$$\int_{\mathcal{W}} |g(w')| \gamma_{w'}(w'; \alpha, w)dw' = \int_{\mathcal{W}(\alpha,w)} |g(w')| \gamma_{w'}(w'; \alpha, w)dw'$$

$$\leq (y^n - y^1) \cdot \sup_{w' \in \mathcal{W}(\alpha,w)} |g(w')| \, \gamma_{w'}(w'; \alpha, w) < \infty$$

for all α and $w \in \mathcal{W}$. Let $\mathscr{D}$ denote the set of continuous functions $g : \mathcal{W} \to \mathbb{R}$ and $\mathscr{D}_\phi$ be the set ϕ-bounded continuous functions, where ϕ is defined in the proof of theorem 3.1. Clearly, $\mathscr{D}_\phi \subseteq \mathscr{D}$.

Under (b), I have that for any ϕ-bounded continuous function $g : \mathcal{W} \to \mathbb{R}$,

$$\int_{\mathcal{W}} |g(w')| \, \gamma_{w'}(w'; \alpha, w) dw' \leq K \int_{\mathcal{W}(\alpha,w)} \phi(w') \gamma_{w'}(w'; \alpha, w) dw' < \infty.$$

For this case, let $\mathscr{D} = \mathscr{D}_\phi$ be the set of ϕ-bounded continuous functions.

Then by assumption D, $P(w, \mathcal{A}; \alpha)$ is $\mathscr{D}$-Lipschitz continuous at α. To see this, notice that for $\alpha' \neq \alpha$ with α' chosen in such a way that either both $c^b(\alpha, w), c^b(\alpha', w) < w$ or $c^b(\alpha, w), c^b(\alpha', w) \geq w$. We can do this, since both $\mathscr{D}$-Lipschitz continuity and $\mathscr{D}$-differentiability need to hold on an open set around α. Then

$$\begin{aligned}
&\left| \int_{\mathcal{W}} g(w') \gamma_{w'}(w'; \alpha, w) dw' - \int_{\mathcal{W}} g(w') \gamma_{w'}(w'; \alpha', w) dw' \right| \\
&\qquad = \left| \int_{\mathcal{W}} g(w') \Big(\gamma_{w'}(w'; \alpha, w) - \gamma_{w'}(w'; \alpha', w) \Big) dw' \right| \\
&\qquad \leq \int_{\mathcal{W}} |g(w')| \, |\gamma_{w'}(w'; \alpha, w) - \gamma_{w'}(w'; \alpha', w)| \, dw'.
\end{aligned}$$

On $\mathcal{W}(\alpha, w) \cap \mathcal{W}(\alpha', w)$ I have that

$$\begin{aligned}
|\gamma_{w'}(w'; \alpha, w) - \gamma_{w'}(w'; \alpha', w)| \leq & M_1(w) \left| R(w - c^b(\alpha, w)) - R(w - c^b(\alpha', w)) \right| \\
= & R \, M_1(w) \left| c^b(\alpha, w) - c^b(\alpha', w) \right| \\
\leq & R \, M_1(w) \Big(|\alpha_0 - \alpha_0'| + w \, |\alpha_1 - \alpha_1'| \Big).
\end{aligned}$$

On the other hand, let

$$\hat{w} = \max \left\{ R(w - c^b(\alpha, w)) + y^n, R(w - c^b(\alpha', w)) + y^n \right\},$$

$$\underline{\hat{w}} = \min \left\{ R(w - c^b(\alpha, w)) + y^n, R(w - c^b(\alpha', w)) + y^n \right\},$$

$$\tilde{w} = \max\left\{R(w - c^b(\alpha, w)) + y^1, R(w - c^b(\alpha', w)) + y^1\right\}$$

$$\underline{\tilde{w}} = \min\left\{R(w - c^b(\alpha, w)) + y^1, R(w - c^b(\alpha', w)) + y^1\right\}$$

so that

$$\begin{aligned}
&\int_{\mathcal{W}\setminus\Big(\mathcal{W}(\alpha,w)\cap\mathcal{W}(\alpha',w)\Big)} |g(w')|\,|\gamma_{w'}(w';\alpha,w) - \gamma_{w'}(w';\alpha',w)|\,dw' \\
=&\int_{\underline{\tilde{w}}}^{\tilde{w}} |g(w')| \max\left\{\gamma_{w'}(w';\alpha,w), \gamma_{w'}(w';\alpha',w)\right\} dw' \\
&+\int_{\underline{\hat{w}}}^{\hat{w}} |g(w')| \max\left\{\gamma_{w'}(w';\alpha,w), \gamma_{w'}(w';\alpha',w)\right\} dw' \\
\leq&\Big(M_2(w) + M_3(w)\Big)\Big(|\alpha_0 - \alpha_0'| + w\,|\alpha_1 - \alpha_1'|\Big).
\end{aligned}$$

So,

$$\begin{aligned}
&\int_{\mathcal{W}} |g(w')|\,|\gamma_{w'}(w';\alpha,w)dw' - \gamma_{w'}(w';\alpha',w)|\,dw' = \\
=&\int_{\mathcal{W}(\alpha,w)\cap\mathcal{W}(\alpha',w)} |g(w')|\,|\gamma_{w'}(w';\alpha,w)dw' - \gamma_{w'}(w';\alpha',w)|\,dw' \\
&+\int_{\mathcal{W}\setminus\Big(\mathcal{W}(\alpha,w)\cap\mathcal{W}(\alpha',w)\Big)} |g(w')|\,|\gamma_{w'}(w';\alpha,w)dw' - \gamma_{w'}(w';\alpha',w)|\,dw' \\
\leq&\Big(K_g\,R\,M_1(w) + M_2(w) + M_3(w)\Big)\Big(|\alpha_0 - \alpha_0'| + w\,|\alpha_1 - \alpha_1'|\Big)
\end{aligned}$$

where $K_g = \int_{\mathcal{W}(\alpha,w)\cap\mathcal{W}(\alpha',w)} |g(w')|\,dw'$.

Similarly, if α, α' and w are such that either $\alpha_0 + \alpha_1 w \leq w \leq \alpha_0' + \alpha_1' w$ or $\alpha_0' + \alpha_1' w \leq w \leq \alpha_0 + \alpha_1 w$, then

$$\begin{aligned}
&\int_{\mathcal{W}} |g(w')|\,|\gamma_{w'}(w';\alpha,w)dw' - \gamma_{w'}(w';\alpha',w)|\,dw' = \\
=&\int_{\mathcal{W}(\alpha,w)\cap\mathcal{W}(\alpha',w)} |g(w')|\,|\gamma_{w'}(w';\alpha,w)dw' - \gamma_{w'}(w';\alpha',w)|\,dw' \\
&+\int_{\mathcal{W}\setminus\Big(\mathcal{W}(\alpha,w)\cap\mathcal{W}(\alpha',w)\Big)} |g(w')|\,|\gamma_{w'}(w';\alpha,w)dw' - \gamma_{w'}(w';\alpha',w)|\,dw' \\
\leq&\Big(K_g\,R\,M_1(w) + M_2(w) + M_3(w)\Big)\Big(|\alpha_0 - \alpha_0'| + w\,|\alpha_1 - \alpha_1'|\Big).
\end{aligned}$$

This follows from a proof similar as before, one just need to notice that if

$\alpha_0 + \alpha_1 w \leq w \leq \alpha_0' + \alpha_1' w$, then

$$0 \leq w(1-\alpha_1) - \alpha_0 \leq (\alpha_0 - \alpha_0') + w(\alpha_1 - \alpha_1'),$$

so that

$$\begin{aligned} \left|c^b(\alpha, w) - c^b(\alpha', w)\right| &= |w(1-\alpha_1) - \alpha_0| \\ &\leq |(\alpha_0 - \alpha_0') + w(\alpha_1 - \alpha_1')| \\ &\leq |\alpha_0 - \alpha_0'| + w\,|\alpha_1 - \alpha_1'|\,, \end{aligned}$$

and similarly for the case when $\alpha_0' + \alpha_1' w \leq w \leq \alpha_0 + \alpha_1 w$.

Step 3: Assumption F implies that

$$\sup_\alpha \|P_\alpha\|_\phi = \sup_\alpha \sup_w \int_{\mathcal{W}} \gamma_{w'}(w'; \alpha, w) \frac{\phi(w')}{\phi(w)} dw' < \infty. \tag{A.4}$$

This result follows directly from our assumption in a similar fashion as the proof of Lemma 4.1 in Heidergott, Hordijk, and Weisshaupt (2006).

Step 4: Since $\phi(\cdot)$ is continuousand the level sets $C_\phi = \{w \mid \phi(w) \leq d\}$ for some $d \in \mathbb{R}_+$ are compact, they are also petite. This implies that $\phi(\cdot)$ is unbounded off petite sets.

Notice that $-R\alpha_0 - (1 - R(1-\alpha_1))w \in \mathscr{D}_\phi$ and I can find a $\lambda \in (0,1)$ such that

$$\sup_w \frac{-R\alpha_0 - (1 - R(1-\alpha_1))w}{\phi(w)} < (\lambda - 1) < 0.$$

If $w \leq \bar{w}$, then

$$\int_{\mathcal{Y}} \phi(y)\gamma_y(y)dy \leq L_1 \leq L_1 + \lambda\phi(w).$$

On the other hand, if $w > \bar{w}$, then

$$\int_{\mathcal{Y}} \phi\Big(R(1-\alpha_1)w - R\alpha_0 + y\Big)\gamma_y(y)dy - \phi(w) =$$

$$\int_{\mathcal{Y}} \phi'(\xi)\Big(R(1-\alpha_1)w - R\alpha_0 + y - w\Big)\gamma_y(y)dy <$$

$$\int_{\mathcal{Y}} y\gamma_y(y)dy - R\alpha_0 - (1 - R(1-\alpha_1))w$$

so that

$$\int_{\mathcal{Y}} \phi\Big(R(1-\alpha_1)w - R\alpha_0 + y\Big)\gamma_y(y)dy \leq$$
$$L_2 + \lambda\phi(w).$$

Let $L = \max\{L_1, L_2\}$, then $L, \lambda, \phi(w)$ satisfy lemma 15.2.8 in Meyn and Tweedie (1993), and by their theorem 16.0.1

$$\|P_\alpha^n - \pi_\alpha\|_\phi \leq K_\alpha \rho_\alpha^n \tag{A.5}$$

for some $K_\alpha < \infty$ and $0 < \rho_\alpha < 1$. But equations (A.4) and (A.5) are the same as in lemma 4.1 and equation (20) in theorem 4.1 in Heidergott, Hordijk, and Weisshaupt (2006), so the proof of their theorem 4.2 holds, and π_α is $\mathscr{D}_\phi$-Lipschitz continuous.

The existence of $\alpha^* = (\alpha_0^*, \alpha_1^*)$ follows now from Weierstrass' theorem.

Additionally, notice that if $g \in \mathscr{D}_\phi$, then

$$\frac{d}{d\alpha_0}\int_{\mathcal{W}} g(w')\gamma_{w'}(w';\alpha,w)dw' = \begin{cases} R\Big(g(R((1-\alpha_1)w-\alpha_0)+y^1)\gamma_y(R((1-\alpha_1)w-\alpha_0)+y^1) - \\ \quad g(R((1-\alpha_1)w-\alpha_0)+y^n)\gamma_y(R((1-\alpha_1)w-\alpha_0)+y^n)\Big) \\ \quad + \int_{\mathcal{W}} g(w')\frac{d}{d\alpha_0}\gamma_{w'}(w';\alpha,w)dw' \qquad \text{if } w \geq \frac{\alpha_0}{1-\alpha_1} \\ 0 \qquad \text{Otherwise} \end{cases}$$

$$\frac{d}{d\alpha_1}\int_{\mathcal{W}} g(w')\gamma_{w'}(w';\alpha,w)dw' = \begin{cases} Rw\Big(g(R((1-\alpha_1)w-\alpha_0)+y^1)\gamma_y(R((1-\alpha_1)w-\alpha_0)+y^1) - \\ \quad g(R((1-\alpha_1)w-\alpha_0)+y^n)\gamma_y(R((1-\alpha_1)w-\alpha_0)+y^n)\Big) \\ \quad + \int_{\mathcal{W}} g(w')\frac{d}{d\alpha_1}\gamma_{w'}(w';\alpha,w)dw' \qquad \text{if } w \geq \frac{\alpha_0}{1-\alpha_1} \\ 0 \qquad \text{Otherwise} \end{cases}$$

are well defined and are the $\mathscr{D}_\phi$-derivatives of $P(w, \mathcal{A}; \alpha)$.

(iii) Follows directly from the the proof of (ii) and the definition of problem (3.9). ∎

Proof of proposition 3.3. (i) By Taylor's theorem

$$
\begin{aligned}
u(c^b(w_t))+\beta E_0 v(R(w_t - c^b(w_t)) + y_t) &= u(c^*(w_t)) + u'(c^*(w_t))(c^b(w_t) - c^*(w_t)) \\
&\quad + u''(\xi_t)(c^b(w_t) - c^*(w_t))^2 + \beta E_0[v(R(w_t - c^*(w_t)) + y_t)] \\
&\quad - \beta R E_0[v'(R(w_t - c^*(w_t)) + y_t)](c^b(w_t) - c^*(w_t)) \\
&\quad + \beta R E_0[v''(\zeta_t)](c^b(w_t) - c^*(w_t))^2 \\
&= v(w_t)+\Big(u'(c^*(w_t)) - \beta R E_0[v'(R(w_t - c^*(w_t)) + y_t)]\Big)(c^b(w_t) - c^*(w_t)) \\
&\quad +\Big(u''(\xi_t) + \beta R E_0[v''(\zeta_t)]\Big)(c^b(w_t) - c^*(w_t))^2 \\
&= v(w_t)+\mu(w_t)(c^b(w_t) - c^*(w_t)) + \Big(u''(\xi_t) + \beta R E_0[v''(\zeta_t)]\Big)(c^b(w_t) - c^*(w_t))^2
\end{aligned}
$$

where $\mu(w_t) \geq 0$ is the Lagrange multiplier associated to the first order condition of (3.3a), $\xi_t \in (c^*(w_t), c^b(w_t))$ and $\zeta_t \in (R(w_t - c^*(w_t)) + y_t, R(w_t - c^b(w_t)) + y_t)$. Thus,

$$
\begin{aligned}
U(\alpha_0, \alpha_1, w_t) - v(w_t) =& u(c^b(w_t)) + \beta E_0[U(\alpha_0, \alpha_1, R(w_t - c^b(w_t)) + y_t)] - v(w_t) \\
=& \mu(w_t)(c^b(w_t) - c^*(w_t)) + \Big(u''(\xi_t) + \beta R E_0[v''(\zeta_t)]\Big)(c^b(w_t) - c^*(w_t))^2 \\
& + \beta E_0[U(\alpha_0, \alpha_1, R(w_t - c^b(w_t)) + y_t) - v(R(w_t - c^b(w_t)) + y_t)].
\end{aligned}
$$

Iterating I get

$$
\begin{aligned}
U(\alpha_0, \alpha_1, w_t) - v(w_t) = & E_0\left[\sum_{j=0}^{T} \big(\mu(w_{t+j})(c^b(w_{t+j}) - c^*(w_{t+j}))\big)\right] \\
& + E_0\left[\sum_{j=0}^{T} \Big(\Big(u''(\xi_{t+j}) + \beta R E_0[v''(\zeta_{t+j})]\Big)(c^b(w_{t+j}) - c^*(w_{t+j}))^2\Big)\right] \\
& + \beta^T E_0[U(\alpha_0, \alpha_1, R(w_{t+T} - c^b(w_{t+T})) + y_{t+T}) - v(R(w_{t+T} - c^b(w_{t+T})) + y_{t+T})]
\end{aligned}
$$

which under our hypothesis gets the desired result.

(ii) Notice that $U(\alpha_0, \alpha_1, w_t) - v(w_t)$ is continuous in w_t and in the stationary distribution $w_t \in [y^1, {}^n w]$. So, $U(\alpha_0, \alpha_1, w_t) - v(w_t) \leq \bar{V}$, for some $0 < \bar{V} < \infty$, and thus, $\lim_{t\to\infty} \beta^t E_0\Big[U(\alpha_0, \alpha_1, w_t) - v(w_t)\Big] = 0$ holds for $w_t \in \mathcal{W}$ π^*-a.e. Replacing the previous equation in the definition of $EV^*_{\alpha_0,\alpha_1} - EV^*$ gives the result.

(iii) Similar as before, but I now make use of the fact that π_{α_0,α_1} is the stationary distribution implied by the rule. ■

Proof of Corollary 3.4. Assume on the contrary that $\alpha_0^* + \alpha_1^* \bar{w} < \bar{w}$, then, since $\mu(w_{t+j}) = 0$ for all $w_{t+j} \geq \bar{w}$, I can find $\alpha \in \Lambda$ such that

$$
\begin{aligned}
c^b(\alpha, w_{t+j}) &= c^*(w_{t+j}) \qquad \forall w_{t+j} \leq \bar{w} \\
c^b(\alpha, \tilde{w}) &= c^*(\tilde{w}) = c^b(\alpha^*, \tilde{w}) \qquad \text{for a particluar } \tilde{w} > \bar{w}
\end{aligned}
$$

for which

$$
\begin{aligned}
(c^b(\alpha, w_{t+j}) - c^*(w_{t+j}))^2 < (c^b(\alpha^*, w_{t+j}) - c^*(w_{t+j}))^2 \qquad \forall w_{t+j} \\
\mu(w_{t+j})(c^b(\alpha, w_{t+j}) - c^*(w_{t+j})) = 0 \qquad \forall w_{t+j}
\end{aligned}
$$

so that

$$
EV^b_{\alpha_0,\alpha_1} - EV^*_b > EV^b_{\alpha_0^*,\alpha_1^*} - EV^*_b
$$

which is a contradiction.

That $\alpha_0^* + \alpha_1^* \bar{w} \geq \bar{w}$ implies that the first part of the integrand in (3.12) is always zero, so that (α_0^*, α_1^*) solves the problem in (3.13). The last equivalence follows from a change of variable and the fact that w_t follows the stationary distribution. ■

Proof of theorem 3.5. From the previous corollary, (α_0^*, α_1^*) is such that $c^b(w) = c^*(w)$ for $w \leq \bar{w}$ and the expected squared difference between $c^b(w)$ and $c^*(w)$ is minimal under π_α. Since,

$$
\begin{aligned}
\beta R E_t u'(c^b(w_{t+1})) - u'(c^b(w_t)) =& \beta R E_t \Big(u'(c^b(w_{t+1})) - u'(c^*(w_{t+1})) \Big) \\
&+ \Big(\beta R E_t u'(c^*(w_{t+1})) - u'(c^*(w_t)) \Big) + \Big(u'(c^*(w_t)) - u'(c^b(w_t)) \Big) \\
=& \beta R E_t \Big(u'(c^b(w_{t+1})) - u'(c^*(w_{t+1})) \Big) + \Big(u'(c^*(w_t)) - u'(c^b(w_t)) \Big) \\
=& \tilde{k}_t (c^b(w_t) - c^*(w_t)),
\end{aligned}
$$

then (α_0^*, α_1^*) must also solve (3.15). If the solution is interior, then the first derivatives with respect to α_0 and α_1 must equal zero, which is condition (3.16), where $\frac{\partial \pi_\alpha}{\partial \alpha_0}$ and $\frac{\partial \pi_\alpha}{\partial \alpha_1}$ are the measured-valued derivatives of π_α defined in Heidergott, Hordijk, and Weisshaupt (2006). ■

Proof theorem 3.10. By assumption, it is the case that $\frac{g_0(\alpha)}{k^0} = \frac{g_1(\alpha)}{k^1}$, so the first order condition can be rewritten as

$$\frac{f_0(\alpha^*)}{k^0} + \frac{g_0(\alpha^*)}{k^0} = 0, \qquad \frac{f_1(\alpha^*)}{k^1} + \frac{g_1(\alpha^*)}{k^1} = 0.$$

This implies

$$\frac{f_0(\alpha^*)}{k^0} = \frac{f_1(\alpha^*)}{k^1},$$

which holds if, and only if, $f_0(\alpha^*) = f_1(\alpha^*) = 0$, i.e. if α^* is a quasi-optimal linear consumption function. In this case, additionally,

$$\frac{g_0(\alpha)}{k^0} = \frac{g_1(\alpha)}{k^1} = 0.$$ ■

Proof of proposition 4.1. Notice that the integrand in (ODE-DG) belongs to $\mathscr{D}_\phi$. So, by a proof similar to that of theorem 3.2 the results of theorem 4.2 of Heidergott, Hordijk, and Weisshaupt (2006) hold, implying that π_{α_0,α_1} is differentiable in both α_0 and α_1. Thus, I can linearize the system around $(\alpha_0^e, \alpha_1^e, M^e)$, obtaining[10]

$$\begin{aligned}
\begin{pmatrix} \frac{d\alpha_0}{\partial\tau} \\ \frac{d\alpha_1}{\partial\tau} \end{pmatrix} &= \nabla h_1(\alpha^e, M^e) \cdot \begin{pmatrix} \alpha_0 - \alpha_0^e \\ \alpha_1 - \alpha_1^e \end{pmatrix} \\
\frac{dM}{d\tau} &= \nabla h_2(\alpha^e, M^e) \cdot \begin{pmatrix} \alpha_0 - \alpha_0^e \\ \alpha_1 - \alpha_1^e \end{pmatrix} - (M - M^e),
\end{aligned}$$

where $\nabla h_2(\alpha^e, M^e)$ is the derivative of the second equation in (ODE-DG) with respect to α_0 and α_1 evaluated at the equilibrium. Since $\nabla h_1(\alpha^e, M^e)$ is Hermitian, all its characteristic roots are real and, by the assumption of negative definiteness, strictly negative. So, (α_0^e, α_1^e) is locally asymptotically stable. But then, as $(\alpha_0, \alpha_1) \to (\alpha_0^e, \alpha_1^e)$, the dynamics of M are determined by the last part alone, which is negative definite, so that $M \to M^e$. ■

[10]Given that the derivative is evaluated at the equilibrium, there is not factor in $(M^e - M)$ in the first equation.

Proof of Theorem 4.3. This theorem is simply an application of (i) proposition 10, (ii) proposition 11, (iii) theorem 13 and (iv) theorem 15 in chapter 1 of part II of Benveniste, Métivier, and Priouret (1990). I only need to show that their assumptions A.1-A.7 hold for this problem.

A.1 This is given by assumption H.

A.2 Follows directly from the definition of $\mathcal{Q}$ and (3.1b).

A.3 In their notation, the function $H(\alpha, M, w)$ is given by the elements after κ_k in our equation (DG), while their function ρ is equal to the zero function in our case. There are four cases to consider:

(a) If $w < \alpha_0 + \alpha_1 w$ and $w' < \alpha_0 + \alpha_1 w'$, then $c(w') = y$.

(b) If $w < \alpha_0 + \alpha_1 w$ and $w' \geq \alpha_0 + \alpha_1 w'$, then $c(w') = \alpha_0 + \alpha_1 y$.

(c) If $w \geq \alpha_0 + \alpha_1 w$ and $w' < \alpha_0 + \alpha_1 w'$, then $c(w') = R(1 - \alpha_1)w - R\alpha_0 + y$.

(d) If $w \geq \alpha_0 + \alpha_1 w$ and $w' \geq \alpha_0 + \alpha_1 w'$, then $c(w') = \alpha_0 + R\alpha_1(1 - \alpha_1)w - R\alpha_0\alpha_1 + \alpha_1 y$.

Let $U_1(w, \alpha) = \big(\beta R E_t u'(c(w')) - u'(\alpha_0 + \alpha_1 w)\big)u''(\alpha_0 + \alpha_1 w)$, then

(a)

$$
\begin{aligned}
|U_1(w,\alpha)| =& \big|\big(\beta R E_t u'(y) - u'(\alpha_0 + \alpha_1 w)\big)u''(\alpha_0 + \alpha_1 w)\big| \\
\leq& \big(|\beta R E_t u'(y)| + |u'(\alpha_0 + \alpha_1 w)|\big)\, |u''(\alpha_0 + \alpha_1 w)| \\
<& \big(\big|\beta R u'(\underline{y})\big| + \big|u'(\alpha_0 + \alpha_1 \underline{y})\big|\big)\, \big|u''(\alpha_0 + \alpha_1 \underline{y})\big| \\
=& \bar{U}_1(\alpha)
\end{aligned}
$$

(b)

$$
\begin{aligned}
|U_1(w,\alpha)| =& \big|\big(\beta R E_t u'(\alpha_0 + \alpha_1 y) - u'(\alpha_0 + \alpha_1 w)\big)u''(\alpha_0 + \alpha_1 w)\big| \\
\leq& \big(|\beta R E_t u'(\alpha_0 + \alpha_1 y)| + |u'(\alpha_0 + \alpha_1 w)|\big)\, |u''(\alpha_0 + \alpha_1 w)|
\end{aligned}
$$

$$
\begin{aligned}
&<\left(\left|\beta R u'(\alpha_0+\alpha_1\underline{y})\right|+\left|u'(\alpha_0+\alpha_1\underline{y})\right|\right)\left|u''(\alpha_0+\alpha_1\underline{y})\right| \\
&=\bar{U}_1(\alpha)
\end{aligned}
$$

(c)

$$
\begin{aligned}
|U_1(w,\alpha)| =& \left|\big(\beta R E_t u'(R(1-\alpha_1)w-R\alpha_0+y)-u'(\alpha_0+\alpha_1 w)\big)u''(\alpha_0+\alpha_1 w)\right| \\
\leq& \big(|\beta R E_t u'(R(1-\alpha_1)w-R\alpha_0+y)|+|u'(\alpha_0+\alpha_1 w)|\big)\,|u''(\alpha_0+\alpha_1 w)| \\
<& \left(\left|\beta R u'(R(1-\alpha_1)\underline{y}-R\alpha_0+\underline{y})\right|+\left|u'(\alpha_0+\alpha_1\underline{y})\right|\right)\left|u''(\alpha_0+\alpha_1\underline{y})\right| \\
=& \bar{U}_1(\alpha)
\end{aligned}
$$

(d)

$$
\begin{aligned}
|U_1(w,\alpha)| =& \left|\big(\beta R E_t u'(\alpha_0+R\alpha_1(1-\alpha_1)w-R\alpha_0\alpha_1+\alpha_1 y)-u'(\alpha_0+\alpha_1 w)\big)u''(\alpha_0+\alpha_1 w)\right| \\
\leq& \big(|\beta R E_t u'(\alpha_0+R\alpha_1(1-\alpha_1)w-R\alpha_0\alpha_1+\alpha_1 y)|+|u'(\alpha_0+\alpha_1 w)|\big)\,|u''(\alpha_0+\alpha_1 w)| \\
<& \left(\left|\beta R u'(\alpha_0+R\alpha_1(1-\alpha_1)\underline{y}-R\alpha_0\alpha_1+\alpha_1\underline{y})\right|+\left|u'(\alpha_0+\alpha_1\underline{y})\right|\right)\left|u''(\alpha_0+\alpha_1\underline{y})\right| \\
=& \bar{U}_1(\alpha)
\end{aligned}
$$

Thus, $|U_1(w,\alpha| \leq \bar{U}_1$, where $\bar{U}_1 = \max_{\text{cases }(a)-(d)} \sup_{(\alpha,M)\in\mathcal{Q}_2} \bar{U}_1(\alpha)$. So,

$$
\left\| M^{-1}\cdot U_1(w,\alpha)\cdot \begin{pmatrix}1\\ w\end{pmatrix}\right\| \leq \left\|M^{-1}\right\|\cdot|U_1(w,\alpha)|\cdot\left\|\begin{pmatrix}1\\ w\end{pmatrix}\right\| \leq K_1\bar{U}_1(1+w^2),
$$

where $K_1 = \sup_{(\alpha,M)\in\mathcal{Q}_2}\|M^{-1}\|$. Similarly, defining

$$
U_2(w,\alpha) = \big(u'(\alpha_0+\alpha_1 w)-\beta R E_t u'(c(w'))\big)u'''(\alpha_0+\alpha_1 w)+\Big(u''(\alpha_0+\alpha_1 w)\Big)^2
$$

one can show that $|U_2(w\alpha)| \leq (\bar{U}_2+K_2)(1+w^2)$, where $\bar{U}_2 = \sup_{(\alpha,M)\in\mathcal{Q}_2}\bar{U}_2(\alpha)$, $\bar{U}_2(\alpha)$ are bounds found in a similar fashion as $\bar{U}_1(\alpha)$, and $K_2 = \sup_{(\alpha,M)\in\mathcal{Q}_2}\|M\|$. So,

$$
\left\|U_2(w,\alpha)\cdot\begin{pmatrix}1 & w\\ w & w^2\end{pmatrix}-M\right\| \leq \|U_2(w,\alpha)\|\cdot\left\|\begin{pmatrix}1 & w\\ w & w^2\end{pmatrix}\right\|+\|M\| \leq (\bar{U}_2+K_2)(1+w^2).
$$

A.4 Their function $h(\alpha, M)$ is given by (ODE-DG). We have that

$$\begin{aligned}
&\left\| \int M^{-1} \cdot U_1(w,\alpha) \cdot \begin{pmatrix} 1 \\ w \end{pmatrix} \pi_\alpha(dw) - \int M'^{-1} \cdot U_1(w,\alpha') \cdot \begin{pmatrix} 1 \\ w \end{pmatrix} \pi_{\alpha'}(dw) \right\| \\
\leq & \left\| \int M^{-1} \cdot U_1(w,\alpha) \cdot \begin{pmatrix} 1 \\ w \end{pmatrix} \pi_\alpha(dw) - \int M'^{-1} \cdot U_1(w,\alpha) \cdot \begin{pmatrix} 1 \\ w \end{pmatrix} \pi_\alpha(dw) \right\| \\
&+ \left\| \int M'^{-1} \cdot U_1(w,\alpha) \cdot \begin{pmatrix} 1 \\ w \end{pmatrix} \pi_\alpha(dw) - \int M'^{-1} \cdot U_1(w,\alpha') \cdot \begin{pmatrix} 1 \\ w \end{pmatrix} \pi_\alpha(dw) \right\| \\
&+ \left\| \int M'^{-1} \cdot U_1(w,\alpha') \cdot \begin{pmatrix} 1 \\ w \end{pmatrix} \pi_\alpha(dw) - \int M'^{-1} \cdot U_1(w,\alpha') \cdot \begin{pmatrix} 1 \\ w \end{pmatrix} \pi_{\alpha'}(dw) \right\| \\
\leq & \bar{U}_1 \int (1+w^2)\pi_\alpha(dw) \cdot \left\| M^{-1} - M'^{-1} \right\| + K_1 \int \left| U_1(w,\alpha) - U_1(w,\alpha') \right| (1+w^2)\pi_\alpha(dw) \\
&+ K_1 \Big(K_3 \left| \alpha_0 - \alpha_0' \right| + K_4 \left| \alpha_1 - \alpha_1' \right| \Big)
\end{aligned}$$

where K_3 and K_4 are given by the Lipschitz continuity of π_α. We only need to show now that $U_1(w, \alpha)$ is Lipschitz continuous. Since

$$\begin{aligned}
\left| u''(\alpha_0 + \alpha_1 w) - u''(\alpha_0' + \alpha_1' w) \right| &= \left| u'''(\xi) \Big((\alpha_0 - \alpha_0') + (\alpha_1 - \alpha_1') w \Big) \right| \\
&\leq \left| u'''(\xi) \right| \Big(\left| \alpha_0 - \alpha_0' \right| + \left| \alpha_1 - \alpha_1' \right| w \Big) \\
&\leq \sup_{\alpha_0, \alpha_1} \left| u'''(\min\left\{ \alpha_0 + \alpha_1 \underline{y}, \underline{y} \right\}) \right| \Big(\left| \alpha_0 - \alpha_0' \right| + \left| \alpha_1 - \alpha_1' \right| w \Big), \\
\left| u'(\alpha_0 + \alpha_1 w) - u'(\alpha_0' + \alpha_1' w) \right| &= \left| u''(\xi') \Big((\alpha_0 - \alpha_0') + (\alpha_1 - \alpha_1') w \Big) \right| \\
&\leq \left| u''(\xi') \right| \Big(\left| \alpha_0 - \alpha_0' \right| + \left| \alpha_1 - \alpha_1' \right| w \Big) \\
&\leq \sup_{\alpha_0, \alpha_1} \left| u''(\min\left\{ \alpha_0 + \alpha_1 \underline{y}, \underline{y} \right\}) \right| \Big(\left| \alpha_0 - \alpha_0' \right| + \left| \alpha_1 - \alpha_1' \right| w \Big), \\
\left| u'(c_{t+1}) - u'(c_{t+1}') \right| &= \left| u''(\xi'')(c_{t+1} - c_{t+1}') \right| \\
&\leq \left| u''(\xi'') \right| \left| c_{t+1} - c_{t+1}' \right| \\
&\leq \sup_{\alpha_0, \alpha_1} \left| u''(\min\left\{ \alpha_0 + \alpha_1 \underline{y}, \underline{y} \right\}) \right| \left| c_{t+1} - c_{t+1}' \right|.
\end{aligned}$$

There are 10 different cases of $\left| c_{t+1} - c_{t+1}' \right|$ to analyze:

(a) If $c_{t+1} = y = c_{t+1}'$, then $\left| c_{t+1} - c_{t+1}' \right| = 0$.

(b) If $c_{t+1} = \alpha_0 + \alpha_1 y$ and $c_{t+1}' = y$, then by assumption, $\alpha_0' + \alpha_1' y > y \geq \alpha_0 + \alpha_1 y$, so that

$$\begin{aligned}\left|c_{t+1}-c'_{t+1}\right| &= |\alpha_0+(\alpha_1-1)y| = |(1-\alpha_1)y-\alpha_0| < |(\alpha'_0-\alpha_0)+(\alpha'_1-\alpha_1)y| \\ &\leq |\alpha_0-\alpha'_0| + |\alpha_1-\alpha'_1|\,\bar{y}.\end{aligned}$$

(c) If $c_{t+1} = R(1-\alpha_1)w - \alpha_0 R + y$ and $c'_{t+1} = y$, then $\alpha'_0 + \alpha'_1 y > y \geq \alpha_0 + \alpha_1 y$, so that

$$\begin{aligned}\left|c_{t+1}-c'_{t+1}\right| &= |R(1-\alpha_1)w-\alpha_0 R| = R\,|(1-\alpha_1)w-\alpha_0| < R\,|(\alpha'_0-\alpha_0)+(\alpha'_1-\alpha_1)w| \\ &\leq R\Big(|\alpha_0-\alpha'_0| + |\alpha_1-\alpha'_1|\,w\Big).\end{aligned}$$

(d) If $c_{t+1} = \alpha_0 + R\alpha_1(1-\alpha_1) - R\alpha_0\alpha_1 + \alpha_1 y$ and $c'_{t+1} = y$, then

$$y < \alpha_0 + \alpha_1 y, \qquad\qquad \alpha'_0 + \alpha'_1 w > w \geq \alpha_0 + \alpha_1 w,$$

and $R(1-\alpha_1)w - \alpha_0 R + y \geq \alpha_0 + R\alpha_1(1-\alpha_1) - R\alpha_0\alpha_1 + \alpha_1 y$, so that

$$\begin{aligned}\left|c_{t+1}-c'_{t+1}\right| &= |\alpha_0 + R\alpha_1(1-\alpha_1) - R\alpha_0\alpha_1 + \alpha_1 y - y| \\ &\leq R\,|(1-\alpha_1)w-\alpha_0| \leq R\Big(|\alpha_0-\alpha'_0| + |\alpha_1-\alpha'_1|\,w\Big).\end{aligned}$$

(e) If $c_{t+1} = \alpha_0 + \alpha_1 y$ and $c'_{t+1} = \alpha'_0 + \alpha'_1 y$, then

$$\left|c_{t+1}-c'_{t+1}\right| = |\alpha_0+\alpha_1 y - \alpha'_0 - \alpha'_1 y| \leq |\alpha_0-\alpha'_0| + |\alpha_1-\alpha'_1|\,\bar{y}.$$

(f) If $c_{t+1} = R(1-\alpha_1)w - \alpha_0 R + y$ and $c'_{t+1} = \alpha'_0 + \alpha'_1 y$, then

$$y \geq \alpha'_0 + \alpha'_1 y, \qquad\qquad \alpha'_0 + \alpha'_1 w > w \geq \alpha_0 + \alpha_1 w,$$

and $\alpha_0 + R\alpha_1(1-\alpha_1) - R\alpha_0\alpha_1 + \alpha_1 y \geq R(1-\alpha_1)w - \alpha_0 R + y$, so that

$$\begin{aligned}\left|c_{t+1}-c'_{t+1}\right| &= |R(1-\alpha_1)w - \alpha_0 R + y - \alpha'_0 - \alpha'_1 y| \\ &= R(1-\alpha_1)w - \alpha_0 R + y - \alpha'_0 - \alpha'_1 y \\ &\leq \alpha_0 + \alpha_1\Big(R(1-\alpha_1)w - \alpha_0 R + y\Big) - \alpha'_0 - \alpha'_1 y \\ &= (\alpha_0-\alpha'_0) + (\alpha_1-\alpha'_1)y + R\alpha_1\Big((1-\alpha_1)w-\alpha_0\Big)\end{aligned}$$

$$
\begin{aligned}
&\leq (\alpha_0 - \alpha_0') + (\alpha_1 - \alpha_1')\bar{y} + R\bar{\alpha}_1\Big((\alpha_1' - \alpha_1)w - (\alpha_0' - \alpha_0)\Big) \\
&\leq (1 + R\bar{\alpha}_1)\left|\alpha_0 - \alpha_0'\right| + (\bar{y} + R\bar{\alpha}_1 w)\left|\alpha_1 - \alpha_1'\right|.
\end{aligned}
$$

(g) If $c_{t+1} = \alpha_0 + R\alpha_1(1-\alpha_1) - R\alpha_0\alpha_1 + \alpha_1 y$ and $c'_{t+1} = \alpha_0' + \alpha_1' y$, then

$$
y \geq \alpha_0' + \alpha_1' y, \qquad\qquad \alpha_0' + \alpha_1' w > w \geq \alpha_0 + \alpha_1 w,
$$

and $\alpha_0 + R\alpha_1(1-\alpha_1) - R\alpha_0\alpha_1 + \alpha_1 y \leq R(1-\alpha_1)w - \alpha_0 R + y$, so that

$$
\begin{aligned}
\left|c_{t+1} - c'_{t+1}\right| &= \left|\alpha_0 + R\alpha_1(1-\alpha_1) - R\alpha_0\alpha_1 + \alpha_1 y - \alpha_0' - \alpha_1' y\right| \\
&= R(1-\alpha_1)w - \alpha_0 R + y - \alpha_0' - \alpha_1' y \\
&\leq \left|\alpha_0 - \alpha_0'\right| + \left|\alpha_1 - \alpha_1'\right|\bar{y} + R\alpha_1\left|(1-\alpha_1)w - \alpha_0\right| \\
&\leq \left|\alpha_0 - \alpha_0'\right| + \left|\alpha_1 - \alpha_1'\right|\bar{y} + R\bar{\alpha}_1\Big(\left|\alpha_0 - \alpha_0'\right| + \left|\alpha_1 - \alpha_1'\right| w\Big) \\
&= (1 + R\bar{\alpha}_1)\left|\alpha_0 - \alpha_0'\right| + (\bar{y} + R\bar{\alpha}_1 w)\left|\alpha_1 - \alpha_1'\right|.
\end{aligned}
$$

(h) If $c_{t+1} = R(1-\alpha_1)w - \alpha_0 R + y$ and $c'_{t+1} = R(1-\alpha_1')w - \alpha_0' R + y$, then

$$
\left|c_{t+1} - c'_{t+1}\right| \leq R\Big(\left|\alpha_0 - \alpha_0'\right| + \left|\alpha_1 - \alpha_1'\right| w\Big).
$$

(i) If $c_{t+1} = R(1-\alpha_1)w - \alpha_0 R + y$ and $c'_{t+1} = \alpha_0' + R\alpha_1'(1-\alpha_1') - R\alpha_0'\alpha_1' + \alpha_1' y$, then

$$
\begin{aligned}
&w \geq \alpha_0 + \alpha_1 w \qquad R(1-\alpha_1)w - \alpha_0 R + y < \alpha_0 + R\alpha_1(1-\alpha_1) - R\alpha_0\alpha_1 + \alpha_1 y \\
&w \geq \alpha_0' + \alpha_1' w \qquad R(1-\alpha_1')w - \alpha_0' R + y \geq \alpha_0' + R\alpha_1'(1-\alpha_1') - R\alpha_0'\alpha_1' + \alpha_1' y
\end{aligned}
$$

so that, if $c_{t+1} - c'_{t+1} \geq 0$, then

$$
\begin{aligned}
\left|c_{t+1} - c'_{t+1}\right| &= R(1-\alpha_1)w - \alpha_0 R + y - \alpha_0' - R\alpha_1'(1-\alpha_1') + R\alpha_0'\alpha_1' - \alpha_1' y \\
&\leq \alpha_0 + R\alpha_1(1-\alpha_1) - R\alpha_0\alpha_1 + \alpha_1 y - \alpha_0' - R\alpha_1'(1-\alpha_1') + R\alpha_0'\alpha_1' - \alpha_1' y \\
&= (\alpha_0 - \alpha_0') + (\alpha_1 - \alpha_1')y + (\alpha_1 - \alpha_1')R\Big((1-\alpha_1)w - \alpha_0\Big) \\
&\quad + \alpha_1' R\Big((\alpha_0' - \alpha_0) + (\alpha_1' - \alpha_1)w\Big) \\
&\leq \left|\alpha_0 - \alpha_0'\right| + \left|\alpha_1 - \alpha_1'\right|\bar{y} + \left|\alpha_1 - \alpha_1'\right| Rw + R\bar{\alpha}_1\Big(\left|\alpha_0' - \alpha_0\right| + \left|\alpha_1' - \alpha_1\right| w\Big)
\end{aligned}
$$

$$=(1+R\bar{\alpha}_1)\left|\alpha_0-\alpha_0'\right|+(\bar{y}+R(1+\bar{\alpha}_1)w)\left|\alpha_1-\alpha_1'\right|.$$

On the other hand, if $c_{t+1}-c_{t+1}'\leq 0$, then

$$\begin{aligned}\left|c_{t+1}'-c_{t+1}\right| =&\alpha_0'+R\alpha_1'(1-\alpha_1')-R\alpha_0'\alpha_1'+\alpha_1'y-R(1-\alpha_1)w+\alpha_0R-y\\ \leq&R(1-\alpha_1')w-\alpha_0'R+y-R(1-\alpha_1)w+\alpha_0R-y\\ \leq&R\Big(\left|\alpha_0-\alpha_0'\right|+\left|\alpha_1-\alpha_1'\right|w\Big).\end{aligned}$$

(j) If $c_{t+1}=\alpha_0+R\alpha_1(1-\alpha_1)-R\alpha_0\alpha_1+\alpha_1y$ and $c_{t+1}'=\alpha_0'+R\alpha_1'(1-\alpha_1')-R\alpha_0'\alpha_1'+\alpha_1'y$, then

$$\begin{aligned}\left|c_{t+1}-c_{t+1}'\right|=&\left|\alpha_0+R\alpha_1(1-\alpha_1)-R\alpha_0\alpha_1+\alpha_1y-\alpha_0'-R\alpha_1'(1-\alpha_1')+R\alpha_0'\alpha_1'-\alpha_1'y\right|\\ \leq&(1+R\bar{\alpha}_1)\left|\alpha_0-\alpha_0'\right|+(\bar{y}+R(1+\bar{\alpha}_1)w)\left|\alpha_1-\alpha_1'\right|.\end{aligned}$$

So, since $R\geq 1$, $\bar{y}>0$ and $\bar{\alpha}_1>0$, I have that in general

$$\left|c_{t+1}-c_{t+1}'\right|\leq(1+R\bar{\alpha}_1)\left|\alpha_0-\alpha_0'\right|+(\bar{y}+R(1+\bar{\alpha}_1)w)\left|\alpha_1-\alpha_1'\right|.$$

This implies that

$$\begin{aligned}\left|U_1(w\alpha)-U_1(w,\alpha')\right|=&\Big|\Big(\beta RE_tu'(c_{t+1})-u'(\alpha_0+\alpha_1w)\Big)u''(\alpha_0+\alpha_1w)\\ &-\Big(\beta RE_tu'(c_{t+1}')-u'(\alpha_0'+\alpha_1'w)\Big)u''(\alpha_0'+\alpha_1'w)\Big|\\ =&\Big|\Big(\beta RE_tu'(c_{t+1})-u'(\alpha_0+\alpha_1w)\Big)\Big(u''(\alpha_0+\alpha_1w)-u''(\alpha_0'+\alpha_1'w)\Big)\\ &+u''(\alpha_0'+\alpha_1'w)\Big(\beta RE_tu'(c_{t+1})-u'(\alpha_0+\alpha_1w)-\beta RE_tu'(c_{t+1}')+u'(\alpha_0'+\alpha_1'w)\Big)\Big|\\ \leq&\left|\beta RE_tu'(c_{t+1})-u'(\alpha_0+\alpha_1w)\right|\left|u''(\alpha_0+\alpha_1w)-u''(\alpha_0'+\alpha_1'w)\right|\\ &+\left|u''(\alpha_0'+\alpha_1'w)\right|\Big(\beta RE_t\left|u'(c_{t+1})-u'(c_{t+1}')\right|+\left|u'(\alpha_0+\alpha_1w)-u'(\alpha_0'+\alpha_1'w)\right|\Big)\\ \leq&(1+\beta R)\sup_{\alpha_0,\alpha_1}\left|u'(\min\left\{\alpha_o+\alpha_1\underline{y},\underline{y}\right\})\right|\sup_{\alpha_0,\alpha_1}\left|u'''(\min\left\{\alpha_0+\alpha_1\underline{y},\underline{y}\right\})\right|\\ &\cdot\Big(\left|\alpha_0-\alpha_0'\right|+\left|\alpha_1-\alpha_1'\right|w\Big)+\sup_{\alpha_0,\alpha_1}\left|u''(\min\left\{\alpha_o+\alpha_1\underline{y},\underline{y}\right\})\right|^2\\ &\cdot\Big(\beta R\Big[(1+R\bar{\alpha}_1)\left|\alpha_0-\alpha_0'\right|+(\bar{y}+R(1+\bar{\alpha}_1)w)\left|\alpha_1-\alpha_1'\right|\Big]\end{aligned}$$

$$
\begin{aligned}
&+ \Big[|\alpha_0 - \alpha_0'| + |\alpha_1 - \alpha_1'|\, w \Big] \Big) \\
=& K_5 \, |\alpha_0 - \alpha_0'| + K_6 \, |\alpha_1 - \alpha_1'| + K_7 \, |\alpha_1 - \alpha_1'| \, w
\end{aligned}
$$

where

$$
\begin{aligned}
K_5 =& (1+\beta R) \sup_{\alpha_0,\alpha_1} \left| u'(\min\left\{\alpha_o + \alpha_1 \underline{y}, \underline{y}\right\}) \right| \sup_{\alpha_0,\alpha_1} \left| u'''(\min\left\{\alpha_0 + \alpha_1 \underline{y}, \underline{y}\right\}) \right| \\
&+ \sup_{\alpha_0,\alpha_1} \left| u''(\min\left\{\alpha_o + \alpha_1 \underline{y}, \underline{y}\right\}) \right|^2 \Big(\beta R (1 + R \bar{\alpha}_1) + 1 \Big), \\
K_6 =& \sup_{\alpha_0,\alpha_1} \left| u''(\min\left\{\alpha_o + \alpha_1 \underline{y}, \underline{y}\right\}) \right|^2 \bar{y}, \\
K_7 =& (1+\beta R) \sup_{\alpha_0,\alpha_1} \left| u'(\min\left\{\alpha_o + \alpha_1 \underline{y}, \underline{y}\right\}) \right| \sup_{\alpha_0,\alpha_1} \left| u'''(\min\left\{\alpha_0 + \alpha_1 \underline{y}, \underline{y}\right\}) \right| \\
&+ \sup_{\alpha_0,\alpha_1} \left| u''(\min\left\{\alpha_o + \alpha_1 \underline{y}, \underline{y}\right\}) \right|^2 \Big(R(1 + \bar{\alpha}_1) + 1 \Big).
\end{aligned}
$$

Thus, I have the Lipschitz continuity of (ODE-DG), i.e. $h(\alpha, M)$.

Now, define $v(\alpha, M, w) = \sum_n (P_\alpha^n - \pi_\alpha) H(\alpha, M, w)$. Let's see that it is well defined. For that, since $\left\| P_\alpha^n - \pi_\alpha \right\|_\phi \leq K_\alpha \rho_\alpha^n$, as was established in the proof of theorem 3.2, I have that $|(P_\alpha^n - \pi_\alpha) H(\alpha, M, w)| \leq K_\alpha \left\| H \right\|_\phi \rho_\alpha^n \phi(w)$. Thus,

$$
\sum_n |(P_\alpha^n - \pi_\alpha) H(\alpha, M, w)| \leq \frac{c_\alpha \left\| H \right\|_\phi}{1 - \rho_\alpha} \phi(w) < \infty.
$$

Furthermore, I have that $(I - \pi_\alpha) v(\alpha, M, w) = H(\alpha, M, w) - h(\alpha, M)$,

$$
|v(\alpha, M, w)| \leq \frac{K_\alpha \left\| H \right\|_\phi}{1 - \rho_\alpha} \phi(w) \leq K_8 (1 + w)
$$

and $P_\alpha v(\alpha, M, w)$ is Lipschitz continuous.

A.5 If $w \leq w^n$, then $w_t \leq w^n$ for all $t \geq 0$, while if $w > w^n$, then $w_t \leq w$, so that

$$
E_{w,\alpha,M}(I((\alpha, M) \in \mathcal{Q}_2,\ k \leq t) \left| w_{t+1} \right|^q) \leq K_8 (1 + w^q),
$$

where $K_8 \geq \max\left\{1, w^n\right\}$.

A.6 This holds by assumption H.

A.7 Corollary 4.2 ensures this. ∎

Proof of theorem 4.5. It suffices to note that if $\alpha = \alpha^q$, then $f_0(\alpha^q) = f_1(\alpha^1) = 0$ in the first order condition (3.16). This implies that (ODE-DG) is equal to zero, i.e. that α^q is an equilibrium of (ODE-DG). Similarly, if α^e is an equilibrium of (ODE-DG), then $f_0(\alpha^q) = f_1(\alpha^1) = 0$, i.e. it is a quasi-optimal linear consumption function. Clearly, under the conditions of theorem 3.10, α^* is quasi-optimal. Finally, notice that in this case the objective function in (3.15) is a strict Lyapunov function for (ODE-DG), thus, α^* is asymptotically stable (Hirsch, Smale, and Devaney, 2004, p. 194-195). ∎