Regeneration under Uncertainty*

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Abstract

In a general one-sector model of stochastic growth where the marginal productivity of capital at zero is finite but can vary widely due to technology shocks, we derive explicitly the limiting optimal propensity to invest (the "slope" of the optimal policy function) at zero. We then derive an explicit condition that ensures it is optimal for capital stocks to regenerate with probability one if they are depleted to levels close enough to zero so that "long run" consumption is strictly positive with probability one. For a widely used class of utility and production functions, a strict violation of our regeneration condition implies almost sure global extinction of capital under the optimal policy. Risk aversion plays an important role in regeneration. Regeneration is likely to be optimal when the degree of risk aversion (near zero) is either high or low, while extinction may be optimal for intermediate levels of risk aversion.

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1 Introduction

A fundamental question in the study of economic growth and dynamic capital accumulation is whether positive consumption levels are sustained in the long run. This acquires particular significance when adverse realizations of exogenous random shocks can reduce productivity of capital, leading to contraction of output, depletion of capital stock and reduction of consumption opportunities. The likelihood with which capital stocks and consumption levels remain positive in the long run i.e., avoid convergence to zero, depends on the probability with which capital stocks regenerate when they are sufficiently depleted; the latter is related to the incentive to invest at small levels of output as well as the manner in which random shocks affect productivity of capital when capital stocks are small. This paper examines the nature of economic fundamentals that create dynamic incentives for regeneration of capital near zero in a stochastic economy. In a general model of optimal economic growth under uncertainty, we provide a fairly tight characterization of the optimal policy function when output is sufficiently small and use this to characterize the economic conditions under which it is optimal to regenerate capital with probability one so as to ensure positive long run consumption; in this context, we specifically focus on the important role of risk aversion.

Our study is motivated by the broad objective of understanding the long run behavior of the economy and its relation to fundamentals. It also acquires significance in view of the concern that social stability may not be compatible with extinction of capital and zero long run consumption, and that intertemporal preferences of economic agents may not adequately capture such social concerns.[1] In applications of the optimal economic growth model to (private and social) management of renewable natural resources whose natural growth is affected by exogenous environmental shocks[2], our analysis provides an understanding of the economics of conservation and extinction of such natural assets. Finally, as avoidance of extinction of capital is a necessary condition for convergence to a *positive* stochastic steady state as well as unbounded expansion or sustained growth of the economy, our analysis contributes to the theoretical understanding of these possibilities.

Specifically, we consider the well known one sector model of discounted stochastic

[1]Perhaps, this is an important motivation for the focus on non-zero steady states and their stability in economic growth theory.

[2]See, Clark (2010).

optimal growth with independent and identically distributed production shocks.[3] We allow for a fairly general class of concave production functions with the restriction that the marginal productivity of capital at zero is finite for all realizations of the random shock.[4] As the behavior of the key economic variables near zero is an important part of our analysis, it is natural to focus on production technologies with finite productivity at zero (and particularly so, for applications of the model to natural resource management). In terms of the productivity of capital when capital stocks are large, we allow for production functions that may exhibit either bounded or unbounded growth. We allow for technology shocks that can cause wide variation in the productivity of capital; in particular, we allow for production functions that may be globally unproductive under adverse shocks. Finally, we allow for a very general class of strictly concave and smooth utility functions that satisfy a mild restriction on relative risk aversion near zero.

As indicated above, the behavior of the stochastic process of capital and consumption near zero depends on the distribution of productivity of capital (near zero) as well as the optimal propensity to invest (or consume) when output is close to zero. The latter captures the role of economic behavior and is directly influenced by the nature of intertemporal preferences including risk aversion and discounting. With the exception of a few well known examples where the utility and production functions have specific functional forms that lead to closed form *linear* solutions for the optimal policy function, the existing literature contains very little by way of a tight characterization of the behavior of the optimal policy function near zero.

The first contribution of this paper is to provide an explicit estimate of the optimal propensity to *invest* as output tends to zero in terms of the discount factor, the distribution of productivity and the degree of relative risk aversion (near zero); while this estimate is derived as a lower bound on the optimal propensity to invest near zero, it is the exact limit of this propensity (as output goes to zero) as long as the optimal *consumption* propensity is bounded away from zero. This explicit characterization of the optimal policy function near zero in a general model (with no specific functional

[3] See, Levhari and Srinivasan (969), Brock and Mirman (1972); Olson and Roy (2006) contains a survey of stochastic optimal growth theory.

[4] The prevalent use of production functions that satisfy the Inada condition (infinite productivity at zero) is largely motivated by technical convenience and in the case of some examples, the ease of obtaining explicit solutions to the dynamic optimization problem. There is not much of economic argument for assuming arbitrarily high return on investment in economies with small capital stock.

forms) is of independent interest and is likely to find many interesting applications apart from those considered in this paper.

Second, we provide a tight condition for almost sure regeneration of capital near zero under the optimal policy. This condition ensures that even if capital stocks are depleted to levels close to zero, they recover with probability one (instead of converging to zero); in consequence, long run optimal consumption, output and capital are strictly positive with probability one. In applications of the stochastic growth model to management of renewable resources, our condition for regeneration ensures that conservation with probability one is optimal. As the optimal investment propensity at zero depends explicitly on risk aversion, risk, discounting and productivity, so does our condition for regeneration.

Third, we obtain an explicit linear upper bound on the optimal investment policy function for a family of widely used utility and production functions. In particular, we consider the class of constant relative risk aversion (CRRA) utility functions and show that for any such utility function there is a class of production functions for which the explicit lower bound on the optimal propensity to invest near zero mentioned earlier is also an *upper* bound on the optimal propensity to invest at any level of output.[5] This result is also of independent interest and will find other interesting applications.

Fourth, for the specific family of utility and production functions mentioned above our general condition for regeneration is tight; a strict violation of the condition implies almost sure convergence to zero of all optimal capital and consumption paths.

Fifth, we explicitly highlight the role of risk aversion in the dynamic incentives for regeneration of capital under uncertainty[6]. We show that if the discount factor and the stochastic production technology are such that the regeneration condition is satisfied when the relative risk aversion (at zero) is 1 (for instance, in the case of log utility function), then regeneration is ensured for *all* admissible utility functions. If the regeneration condition is strictly violated when the relative risk aversion at zero is 1, almost sure regeneration is still ensured if relative risk aversion is either small enough or large enough, but almost sure extinction may be optimal when relative risk

[5]It is well known that with CRRA utility functions, the optimal policy function is linear when the production function is linear; our (linear) upper bound on the optimal investment function generalizes this to a class of possibly nonlinear production functions.

[6]For an early analysis of the comparative dynamics of the curvature of the utility function, see Danthine and Donaldson (1981). On a somewhat different note, Jones et al (2004) show that the qualitative relationship between volatility and "mean" growth depends on the curvature of the utility function.

aversion is in an intermediate range (close to 1). This demonstrates the non-monotone effect of increase in risk aversion on regeneration/extinction; increased curvature of the utility function increases the incentive to smoothen intertemporal consumption which makes the economy move away from extinction paths but at the same time, higher risk aversion increases the incentive to favor certainty of current consumption against the uncertainty of future consumption that works against the incentive to accumulate and may push the economy towards extinction.

As mentioned above, regeneration near zero is necessary for convergence (in distribution) of capital and consumption processes to a positive stochastic steady state[7]. Existence and stability of non-zero steady states is a key topic in economic growth theory. In line with this, the early literature in stochastic growth theory focuses on stochastic steady states whose support is *bounded away* from zero and therefore makes assumptions to ensure that when output is close to zero, it is optimal to expand output and capital even under the worst realization of the exogenous shock[8]. This approach is somewhat restrictive as it rules out a wide class of economic environments where for small levels of output, it may be optimal to allow capital and output to decline under "bad" realizations of the shock and expand when the realizations are "good"; indeed, almost sure expansion near zero is not even a technologically feasible option if marginal productivity of capital at zero is less than one for certain shocks.[9]

Over the last two decades, a growing literature in stochastic growth theory has extended the model to allow for production technologies where depending on the realization of the exogenous shock, the output resulting from any level of capital input may be arbitrarily small or large. Definitive conditions for global stability and positive steady state in this framework are outlined in Stachurski (2002), Nishimura and Stachurski (2005) and Kamihigashi (2007); some of their key conditions are geared towards ensuring almost sure regeneration near zero; these conditions involve restrictions on the stochastic technology and discounting, but they do not involve the utility

[7] An invariant distribution that puts no probability mass on zero.

[8] Brock and Mirman (1972) and several other early papers ensure this by assuming infinite marginal productivity at zero and strictly positive probability mass on the worst production shock. See also, Mirman and Zilcha (1975), Brock and Majumdar (1978), Majumdar, Mitra and Nyarko (1989), Olson (1989) and Hopenhayn and Prescott (1992). Less extreme conditions that involve joint restrictions on preferences and technology are contained in Olson and Roy (2000), Mitra and Roy (2006, 2012).

[9] Optimal paths may not be bounded away from zero even if the technology is assumed to be infinitely productive near zero with probability one; see Mirman and Zilcha (1976) and Mitra and Roy (2012).

function or risk preference[10]. Our condition for regeneration is weaker than these conditions and explicitly involves risk aversion or the curvature of the utility function. While it is not our intention to focus on the issue of convergence to a stochastic steady state in this paper, our condition for regeneration near zero can be useful in weakening the conditions for a globally stable positive steady state; we outline such a result for the case of bounded growth technology using the characterization by Kamihigashi and Stachurski (2014) of global stability of monotone stochastic processes.

Section 2 outlines the model and some basic results. Section 3 contains our key result characterizing the optimal propensity to invest near zero. Section 4 outlines our condition for almost sure regeneration near zero. Section 5 outlines an upper bound on the optimal investment policy function for a specific family of utility and production functions and uses this to illustrate the tightness of our condition for regeneration. Section 6 discusses the effect of change in relative risk aversion on the optimality of regeneration and extinction. Section 7 contains an informal discussion on how our condition for regeneration can be used to ensure global convergence to a positive stochastic steady state; the formal results pertaining to this are contained in the Appendix.

2 Model

We consider an infinite horizon one-good representative agent economy. Let $\mathbb{N}$ denote the set of natural numbers $\{0, 1, 2, ...\}$ and $\mathbb{N}_+$ the set of strictly positive natural numbers; let $\mathbb{R}_+$ and $\mathbb{R}_{++}$ denote respectively the sets of non-negative and strictly positive real numbers. Time is discrete and is indexed by $t \in \mathbb{N}$. The initial stock of output $y_0 \in \mathbb{R}_+$ is given. At each date t, the representative agent observes the current stock of output $y_t \in \mathbb{R}_+$ and chooses the level of current investment x_t, and the current consumption level c_t, such that

$$c_t \geq 0, x_t \geq 0, c_t + x_t \leq y_t$$

This generates y_{t+1}, the output next period through the relation

$$y_{t+1} = f(x_t, r_{t+1})$$

[10]See also, Nishimura et al (2006).

where f is the "aggregate" production function and r_{t+1} is a random production shock realized at the beginning of period $(t+1)$.

2.1 Production

The following assumption is made on the sequence of random shocks:

(R.1) $\{r_t\}_{t=1}^{\infty}$ is an independent and identically distributed random process defined on a probability space $(\Omega, \mathcal{F}, P)$, where the marginal distribution function is denoted by Ψ. The support of this distribution is a non-degenerate set $A \subset \mathbb{R}_{++}$.

The production function $f : \mathbb{R}_+ \times A \to \mathbb{R}_+$ is assumed to satisfy the following standard monotonicity, concavity, measurability and smoothness restrictions on the production function:

(T.1) Given any $r \in A$, $f(., r)$ is assumed to be continuously differentiable and concave on $\mathbb{R}_+$, with $f(0, r) = 0$; further, $f'(\cdot, r) = \frac{\partial f(x,r)}{\partial x} > 0$ on $\mathbb{R}_+$. For any $x \geq 0$, $f(x, .) : A \to \mathbb{R}_+$ is a (Borel) measurable function..

Note that **(T.1)** implies that

$$f'(0, r) = \lim_{x \downarrow 0} f'(x, r) < \infty$$

i.e., for each realization of the random shock, marginal productivity is bounded. In other words, we do not allow for production functions where the Inada condition holds at zero.

For each $r \in A$, let $B(r)$ denote the marginal product at zero investment:

$$B(r) = f'(0, r).$$

Observe that for all $x \geq 0$,

$$f(x, r) \leq B(r)x \tag{1}$$

We assume that

(T.2)

$$\underline{B} = \inf_{r \in A} B(r) > 0, \ \overline{B} = \sup_{r \in A} B(r) < \infty. \tag{2}$$

Note that **(T.2)** allows for production technologies that are unproductive at all levels of capital input.

For any investment level $x \geq 0$, let the upper and lower bound of the support of output next period be denoted by $\overline{f}(x)$ and $\underline{f}(x)$, respectively. In particular,

$$\overline{f}(x) = \sup_{r \in A} f(x,r), \underline{f}(x) = \inf_{r \in A} f(x,r). \tag{3}$$

It is easy to check that $\underline{f}(.)$ and $\overline{f}(x)$ are non-decreasing on $\mathbb{R}_+$, $\underline{f}(0) = \overline{f}(0) = 0$ and $\underline{f}(.)$ is concave on $\mathbb{R}_+$. Further, **(T.1)** and **(T.2)** imply that

$$0 < \underline{f}(x) \leq \overline{f}(x) < \infty \tag{4}$$

for all $x > 0$.[11]

2.2 Preferences

We denote by u the one period utility function from consumption and we assume that:

(U.1) $u : \mathbb{R}_+ \to \mathbb{R} \cup \{-\infty\}$ is twice continuously differentiable on $\mathbb{R}_{++}$, $u'(c) > 0, u''(c) < 0$ for all $c > 0$.

(**U.2**) $\lim_{c \to 0} u(c) = u(0)$; $\lim_{c \to 0} u'(c) = +\infty$.

Note that we allow for unbounded utility functions. For $c > 0$, let the Arrow-Pratt measure of relative risk aversion at c be defined by:

$$\rho(c) = -\frac{u''(c)c}{u'(c)}$$

We assume that $\rho(c)$ converges to a strictly positive number as $c \to 0$:

(**U.3**)

$$\lim_{c \to 0} \rho(c) = \rho_0 > 0$$

2.3 The Optimization Problem

Given an initial stock $y \in \mathbb{R}_+$, a stochastic process $\{y_t, c_t, x_t\}$ is *feasible* from y if it satisfies $y_0 = y$, and:

$$(i)\ c_t \geq 0, x_t \geq 0, c_t + x_t \leq y_t, \text{ for all } t \in \mathbb{N}$$
$$(ii)\ y_t = f(x_{t-1}, r_t) \text{ for } t \in \mathbb{N}_+$$

[11] Thus, we do not allow for "unbounded shocks" though we allow the allow the output resulting from any level of capital input to vary widely with the shock.

and (iii) for each $t \in \mathbb{N}$, $\{c_t, x_t\}$ are $\mathcal{F}_t$ adapted where $\mathcal{F}_t$ is the (sub) σ-field generated by partial history from periods 0 through t.

Let $\delta \in (0,1)$ denote the time discount factor. The objective of the representative agent is to maximize the expected value of the discounted sum of utilities from consumption:

$$E\left[\sum_{t=0}^{\infty} \delta^t u(c_t)\right]$$

Given $y \geq 0$, define the stochastic process of consumption $\{c_t^M\}$ by: $c_0^M = y$, $c_{t+1}^M = f(c_t^M, r_{t+1})$ for all $t \geq 0$. Thus, c_t^M is an upper bound on feasible consumption in period t. We assume that:

(D.1) For all $y \geq 0$,

$$E\left[\sum_{t=0}^{\infty} \delta^t u(c_t^M)_+\right] < \infty$$

where $u(c)_+ = \max\{u(c), 0\}$.

Assumption (**D.1**) ensures that for any feasible stochastic process $\{y_t, c_t, x_t\}$ from $y \geq 0$, the objective of the representative agent

$$E\left[\sum_{t=0}^{\infty} \delta^t u(c_t))\right]$$

is well defined though it may equal $-\infty$, and that (see, Kamihigashi 2007)

$$E\left[\sum_{t=0}^{\infty} \delta^t u(c_t))\right] = \sum_{t=0}^{\infty} \delta^t E[u(c_t)] \tag{5}$$

Note that (**D.1**) is always satisfied if either u is bounded above or alternatively, if $\text{limsup}_{x\to\infty}[\overline{f}(x)/x] < 1$ i.e., the technology exhibits bounded growth.

Given initial stock $\overline{y} \geq 0$, a feasible stochastic process $\{y_t, c_t, x_t\}$ is *optimal* from $\overline{y}$ if for every feasible stochastic process $\{y_t', c_t', x_t'\}$ from $\overline{y}$,

$$E\left[\sum_{t=0}^{\infty} \delta^t u(c_t))\right] \geq E\left[\sum_{t=0}^{\infty} \delta^t u(c_t')\right]$$

For $y \geq 0$, let $V(y)$, the value function be defined by

$$V(y) = \sup\left\{ E\sum_{t=0}^{\infty}\delta^t u(c_t)) : \{c_t, x_t, y_t\} \text{ is a feasible stochastic process from } y \right\}$$

We assume:

(**D.2**) $V(y) > -\infty$ for all $y > 0$.

Note that (**D.2**) is always satisfied if $u(0) > -\infty$ or alternatively, if $\underline{B} > 1$ i.e., the worst case production function is productive near zero; if neither of these hold, it is satisfied under some restrictions on the discount factor δ.

Combined with assumption (**D.1**), we now have

$$-\infty < V(y) < +\infty, \forall y > 0$$

A *consumption (policy) function,* is a function $\widetilde{c} : \mathbb{R}_+ \to \mathbb{R}_+$, satisfying:

$$0 \leq \widetilde{c}(y) \leq y \text{ for all } y \in \mathbb{R}_+$$

Note that this implies $\widetilde{c}(0) = 0$. Associated with a consumption function $\widetilde{c}(\cdot)$, is an *investment (policy) function* $\widetilde{x} : \mathbb{R}_+ \to \mathbb{R}$, defined by

$$\widetilde{x}(y) = y - \widetilde{c}(y) \text{ for all } y \in \mathbb{R}_+$$

Thus, the investment function $\widetilde{x}(\cdot)$ satisfies:

$$0 \leq \widetilde{x}(y) \leq y \text{ for all } y \in \mathbb{R}_+$$

A feasible stochastic process $\{y_t, c_t, x_t\}$ is said to be *generated by* a consumption function $\widetilde{c}(y)$ from initial stock $\overline{y} \in \mathbb{R}_+$

$$\begin{aligned} y_0 &= \overline{y};\ y_{t+1} = f(y_t - \widetilde{c}(y_t), r_{t+1}) \text{ for } t \geq 0; \\ c_t &= \widetilde{c}(y_t),\ x_t = y_t - \widetilde{c}(y_t) \text{ for } t \geq 0. \end{aligned}$$

A consumption (policy) function $c(y)$ is said to be optimal if for every initial stock $\overline{y} \in \mathbb{R}_+$, the stochastic process $\{y_t, c_t, x_t\}$ generated by the function $c(.)$ is optimal; we refer to the investment policy function $x(y) = y - c(y)$ as the optimal investment function.

Standard dynamic programming arguments (see Theorem 2.1 in Kamihigashi, 2007) imply:

Lemma 1 *The value function $V(y)$ satisfies the functional equation:*

$$V(y) = \max_{0\leq c\leq y}[u(c) + \delta E[V(f(y-c,r)]]. \tag{6}$$

$V(y)$ is continuous, strictly increasing and strictly concave on $\mathbb{R}_{++}$. For each $y \geq 0$, the maximization problem on the right hand side of (6) has a unique solution $c(y)$ and the consumption (policy) function $c(.)$ is the unique optimal consumption (policy) function. For all $y > 0$, $c(y) > 0$ and $x(y) = y-c(y) > 0$. $x(y)$ and $c(y)$ are continuous and strictly increasing in y on $\mathbb{R}_+$. For all $y > 0$,the following Ramsey-Euler equation holds:

$$\begin{aligned} u'(c(y)) &= \delta E[u'(c(f(x(y),r)))f'(x(y),r)] \\ &= \delta \int_A u'(c(f(x(y),r)))f'(x(y),r)d\Psi(r) \end{aligned} \tag{7}$$

Let $H : \mathbb{R}_+ \times A$ be *the optimal transition function* defined by:

$$H(y,r) = f(x(y),r).$$

For $y \in \mathbb{R}_+$, the optimal stochastic process of output, consumption and investment $\{y_t(y), c_t(y), x_t(y)\}$ (generated by the optimal policy function) from initial stock y are given by

$$\begin{aligned} y_0(y) &= y,\ y_{t+1}(y) = H(y_t(y), r_{t+1}), t \in \mathbb{N} \\ c_t(y) &= c(y_t(y)), x_t(y) = x(y_t(y)), t \in \mathbb{N} \end{aligned}$$

Let $\overline{H} : \mathbb{R}_+ \to \mathbb{R}_+$ and $\underline{H} : \mathbb{R}_+ \to \mathbb{R}_+$ denote the upper and lower envelope of the transition functions defined by:

$$\overline{H}(y) = \overline{f}(x(y)), \underline{H}(y) = \underline{f}(x(y))$$

As $\underline{f}(.)$ is nondecreasing and $x(y)$ is increasing, $\underline{H}(.)$ is non-decreasing on $\mathbb{R}_+$.

3 Optimal Propensity to Invest

The behavior of the optimal policy function near zero is of crucial importance in determining the possibility of regeneration of capital when the capital stock is sufficiently depleted. In this section, we provide an explicit characterization of the limiting optimal propensity to invest as output converges to zero in terms the discount factor, the probability distribution of marginal productivity of capital as investment goes to zero and the degree of relative risk aversion as consumption goes to zero.

3.1 Main Result

Recall, that $\rho(c) = -\frac{u''(c)c}{u'(c)}$ is the Arrow-Pratt measure of relative risk aversion at $c > 0$. Under assumption (**U.3**), $\rho(c) \to \rho_0 > 0$ as $c \to 0$; thus ρ_0 is the (limiting) risk aversion at zero. Also, recall that $B(r) = f'(0, r)$ is the marginal product at zero investment corresponding to realization r of the productivity shock.

Define

$$s_0 = \left(\delta E((B(r))^{1-\rho_0}\right)^{1/\rho_0} = \left(\delta \int_A (B(r))^{1-\rho_0} d\Psi(r)\right)^{1/\rho_0} \tag{8}$$

Assumption (**T.2**) ensures that s_0 is well defined and $0 < s_0 < \infty$.

Finally, define $\theta \in (0, 1]$ by

$$\theta = \min\{s_0, 1\} \tag{9}$$

The main result we establish in this section is as follows:

Proposition 1 *(i)*

$$\liminf_{y \to 0} \frac{x(y)}{y} \geq \theta = \min\{s_0, 1\} \tag{10}$$

(ii) If the optimal propensity to invest $\frac{x(y)}{y}$ *is bounded away from* 1 *as* $y \to 0$, *then* $s_0 < 1$ *and*

$$\lim_{y \to 0} \frac{x(y)}{y} = s_0 = \theta$$

Part (i) of Proposition 1 provides an explicit lower bound θ for the optimal propensity to invest as output converges to zero. This lower bound does not require the production function or the utility function to have any specific functional form. Our subsequent analysis of regeneration of capital near zero will use this lower bound.

Note that (10) implies that if $s_0 \geq 1$, then $\frac{x(y)}{y} \to 1$ as $y \to 0$. Part (ii) of the proposition indicates that the lower bound θ is "tight" in the sense that it is the exact limit of the optimal propensity to invest as output converges to zero (loosely, the slope of the policy function at zero) if the optimal propensity to invest is bounded away from 1 i.e., if the optimal propensity to consume $\frac{c(y)}{y}$ is bounded away from zero.[12]

3.2 Proof of Proposition 1

We begin by stating a useful result reported in Mitra and Roy (2012, Lemma 4):

Lemma 2 *(Mitra and Roy 2012) For any $c^1 > 0, c^2 > 0,\ c^2 \geq c^1$, if $\rho(c) \in [\underline{\rho}, \overline{\rho}] \subset \mathbb{R}_{++}$ for all $c \in [c^1, c^2]$, then*

$$\left(\frac{c^2}{c^1}\right)^{-\overline{\rho}} \leq \frac{u'(c^2)}{u'(c^1)} \leq \left(\frac{c^2}{c^1}\right)^{-\underline{\rho}}. \tag{11}$$

Next, we establish some bounds on the limiting behavior of the propensity to consume as output tends to zero. The following lemma shows that the optimal propensity to consume is bounded away from one (i.e., the optimal propensity to invest is bounded away from zero); assumption (**U.3**) which ensures that relative risk aversion is bounded away from zero plays an important role in this result.

Lemma 3

$$\lim_{y \to 0} \sup \frac{c(y)}{y} < 1. \tag{12}$$

Proof. Suppose to the contrary that $\lim_{y\to 0} \sup \frac{c(y)}{y} = 1$.Fix $\gamma \in (0, 1)$.There exists $\widetilde{y} > 0$ such that $\rho(c) \geq \gamma\rho_0$ for all $c \in (0, \overline{f}(\widetilde{y})]$. Choose $\eta \in (1 - \frac{1}{\overline{B}}, 1)$. Then, $\overline{f}((1-\eta)y) \leq \overline{B}(1-\eta)y < y$ for all $y \in (0, \widetilde{y}]$. As $\lim_{y\to 0} \sup \frac{c(y)}{y} = 1$, there exists a sequence $\{y^n\}$ converging to zero, $y^n \in (0, \widetilde{y})$ for all n such that $\frac{c(y^n)}{y^n} \geq \eta$ for all n.Then, $f(x(y^n), r) \leq \overline{f}(x(y^n)) \leq \overline{f}((1-\eta)y^n) < y^n$ for all $r \in A$. From the

[12] It is worth noting that if the production function is linear i.e., $f(x, r) = rx$, the utility function exhibits constant relative risk aversion and $\theta < 1$, then the optimal investment policy function is given by $x(y) = \theta y$ i.e., θ is the optimal propensity to invest at all levels of output. See, for instance, de Hek (1999) and de Hek and Roy (2001).

Ramsey-Euler equation (7):

$$\begin{aligned}
\frac{1}{\delta} &= E\left[\frac{u'(c(f(x(y^n),r)))}{u'(c(y^n))} f'(x(y^n),r)\right] \\
&\geq E\left[\left(\frac{c(f(x(y^n),r))}{c(y^n)}\right)^{-\gamma\rho_0} f'(x(y^n),r)\right], \text{ using Lemma 2} \\
&= E\left[\left(\frac{c(f(x(y^n),r))}{f(x(y^n),r)}\right)^{-\gamma\rho_0} \left(\frac{f(x(y^n),r)}{x(y^n)}\right)^{-\gamma\rho_0} f'(x(y^n),r))\right] \left(\frac{\frac{c(y^n)}{y^n}}{1-\frac{c(y^n)}{y^n}}\right)^{\gamma\rho_0} \\
&\geq E\left[\left(\frac{f(x(y^n),r)}{x(y^n)}\right)^{-\gamma\rho_0} f'(x(y^n),r))\right] \left(\frac{\eta}{1-\eta}\right)^{\gamma\rho_0},
\end{aligned}$$

and taking the liminf as $n \to \infty$ and using Fatou's Lemma (see, for instance, Section 4.3.3 in Dudley, 2002) we have

$$\begin{aligned}
\frac{1}{\delta} &\geq E\left[\liminf_{n\to\infty} \left(\frac{f(x(y^n),r)}{x(y^n)}\right)^{-\gamma\rho_0} f'(x(y^n),r)\right] \left(\frac{\eta}{1-\eta}\right)^{\gamma\rho_0} \\
&= E\left[(B(r))^{1-\gamma\rho_0}\right] \left(\frac{\eta}{1-\eta}\right)^{\gamma\rho_0}
\end{aligned}$$

so that

$$\frac{\eta}{1-\eta} \leq \left(\frac{1}{\delta E\left[(B(r))^{1-\gamma\rho_0}\right]}\right)^{\frac{1}{\gamma\rho_0}}$$

As $\rho_0 > 0$ and the right hand side of the above inequality is independent of η, we have a contradiction for η close enough to 1. ■

The next lemma establishes an upper bound on the limiting optimal propensity to consume at zero under the assumption that it is strictly positive.

Lemma 4 *Suppose that*

$$\lim_{y\to 0}\sup \frac{c(y)}{y} > 0.$$

Then, $s_0 < 1$ *and*

$$\lim_{y\to 0}\sup \frac{c(y)}{y} \leq 1 - s_0$$

Proof. Let $\overline{z} = \lim_{y\to 0}\sup \frac{c(y)}{y} > 0$. Using Lemma 3, $0 < \overline{z} < 1$. We will now show that

$$\overline{z} \leq 1 - s(\rho_0) \tag{13}$$

Fix $\lambda \in (0,1)$ and $\widehat{M} > 0$ such that $\widehat{M} < \frac{\overline{z}}{(1-\overline{z})\overline{B}}$. There exists $\overline{h} \in (0, \min\{\overline{z}, 1-\overline{z}\})$ such that

$$\widehat{M} \leq \frac{(\overline{z}-h)}{(1-(\overline{z}-h))\overline{B}} \text{ for all } h \in (0,\overline{h}) \tag{14}$$

Choose any ϵ, h such that $0 < \epsilon < \rho_0, h \in (0,\overline{h})$. There exists $\overline{y} > 0$ such that

$$\rho_0 - \epsilon \leq \rho(c) \leq \rho_0 + \epsilon \text{ for all } c \in (0, \overline{f}(\overline{y})). \tag{15}$$

and

$$f'(\overline{y}, r) \geq \lambda \underline{B} \text{ for all } r \in A \tag{16}$$

By definition of $\overline{z}$ there exists a sequence $\{z^n\}_{n=1}^{\infty}$, $z^n \in (0,\overline{y})$ for all n, $z^n \to 0$ as $n \to \infty$,$\{\frac{c(z^n)}{z^n}\}$ is convergent

$$\overline{z} + h \geq \frac{c(z^n)}{z^n} \geq \overline{z} - h \text{ for all } n. \tag{17}$$

From the Ramsey-Euler equation (7) and using Lemma 2 and (15), we have:

$$\begin{aligned}
\frac{1}{\delta} &= E\left[\frac{u'(c(f(x(z^n),r)))}{u'(c(z^n))} f'(x(z^n),r)\right] \\
&\geq E\left[\left(\frac{c(f(x(z^n),r))}{c(z^n)}\right)^{-(\rho_0+\epsilon)} f'(x(z^n),r) I_{[f(x(z^n),r)\geq z^n]}\right] \\
&\quad + E\left[\left(\frac{c(f(x(z^n),r))}{c(z^n)}\right)^{-(\rho_0-\epsilon)} f'(x(z^n),r) I_{[f(x(z^n),r)< z^n]}\right]
\end{aligned} \tag{18}$$

Observe that if $f(x(z^n),r) \geq z^n$, then

$$\begin{aligned}
&\left(\frac{c(f(x(z^n),r))}{c(z^n)}\right)^{-(\rho_0+\epsilon)} \\
= &\left(\frac{x(z^n)}{c(z^n)}\right)^{-(\rho_0+\epsilon)} \left(\frac{c(f(x(z^n),r))}{f(x(z^n),r)}\right)^{-(\rho_0+\epsilon)} \left(\frac{f(x(z^n),r)}{x(z^n)}\right)^{-(\rho_0+\epsilon)} \\
\geq &\left(\frac{x(z^n)}{c(z^n)}\right)^{-\rho_0} \left(\frac{c(f(x(z^n),r))}{f(x(z^n),r)}\right)^{-\rho_0} \left(\frac{f(x(z^n),r)}{x(z^n)}\right)^{-\rho_0} \left(\frac{1-(\overline{z}-h)}{(\overline{z}-h)}\overline{B}\right)^{-\epsilon} \text{(using (17))} \\
\geq &\left(\frac{x(z^n)}{c(z^n)}\right)^{-\rho_0} \left(\frac{c(f(x(z^n),r))}{f(x(z^n),r)} \frac{f(x(z^n),r)}{x(z^n)}\right)^{-\rho_0} (\widehat{M})^{\epsilon} \text{ (using (14))}
\end{aligned}$$

so that

$$\begin{aligned}
& E\left[\left(\frac{c(f(x(z^n),r))}{c(z^n)}\right)^{-(\rho_0+\epsilon)} f'(x(z^n),r) I_{[f(x(z^n),r)\geq z^n]}\right] \\
\geq & \left(\widehat{M}\right)^{\epsilon} \left(\frac{x(z^n)}{c(z^n)}\right)^{-\rho_0} \bullet \\
& \bullet E\left(\left(\frac{c(f(x(y^n),r))}{f(x(y^n),r)} \frac{f(x(z^n),r)}{x(z^n)}\right)^{-\rho_0} f'(x(z^n),r) I_{[f(x(z^n),r)\geq z^n]}\right) \qquad (19)
\end{aligned}$$

If $f(x(z^n),r) < z^n$, then

$$\begin{aligned}
\left(\frac{c(f(x(z^n),r))}{c(z^n)}\right)^{-(\rho_0+\epsilon)} & \geq \left(\frac{c(f(x(z^n),r))}{c(z^n)}\right)^{-\rho_0} \\
= & \left(\frac{x(z^n)}{c(z^n)}\right)^{-\rho_0} \left(\frac{c(f(x(z^n),r))}{f(x(z^n),r)} \frac{f(x(z^n),r)}{x(z^n)}\right)^{-\rho_0}
\end{aligned}$$

so that

$$\begin{aligned}
& E\left[\left(\frac{c(f(x(z^n),r))}{c(z^n)}\right)^{-(\rho_0-\epsilon)} f'(x(z^n),r) I_{[f(x(z^n),r)<z^n]}\right] \\
\geq & E\left[\left(\frac{c(f(x(z^n),r))}{f(x(z^n),r)} \frac{f(x(z^n),r)}{x(z^n)}\right)^{-\rho_0} f'(x(z^n),r) I_{[f(x(z^n),r)<z^n]}\right] \left(\frac{x(z^n)}{c(z^n)}\right)^{-\rho_0} \qquad (20)
\end{aligned}$$

Using, (18), (19) and (20), we have

$$\frac{1}{\delta} \geq \left(\frac{x(z^n)}{c(z^n)}\right)^{-\rho_0} E\left[\left(\frac{c(f(x(z^n),r))}{f(x(z^n),r)} \frac{f(x(z^n),r)}{x(z^n)}\right)^{-\rho_0} f'(x(z^n),r)\right] \min\left\{\widehat{M}^{\epsilon}, 1\right\} \qquad (21)$$

which implies:

$$\begin{aligned}
& \left(\frac{c(z^n)}{z^n}\right)^{-\rho_0} \left(\frac{x(z^n)}{z^n}\right)^{\rho_0} \\
\geq & \delta E\left[\left(\frac{c(f(x(z^n),r))}{f(x(z^n),r)}\right)^{-\rho_0} \left(\frac{f(x(z^n),r)}{x(z^n)}\right)^{-\rho_0} f'(x(z^n),r)\right] \min\left\{\widehat{M}^{\epsilon}, 1\right\} \qquad (22)
\end{aligned}$$

For each $r \in A$,

$$\lim_{n\to\infty}\inf\left[\left(\frac{c(f(x(z^n),r))}{f(x(z^n),r)}\right)^{-\rho_0}\left(\frac{f(x(z^n),r)}{x(z^n)}\right)^{-\rho_0} f'(x(z^n),r)\right] \geq \overline{z}^{-\rho_0}(B(r))^{1-\rho_0} \tag{23}$$

Taking the liminf as $n \to \infty$ on both sides of (22) and using Fatou's lemma:

$$\begin{aligned}
& \lim_{n\to\infty}\inf \frac{c(z^n)}{z^n}^{-\rho_0}\left(\frac{x(z^n)}{z^n}\right)^{\rho_0} \\
\geq \quad & \delta E\left[\lim_{n\to\infty}\inf\left(\frac{c(f(x(z^n),r))}{f(x(z^n),r)}\frac{f(x(z^n),r)}{x(z^n)}\right)^{-\rho_0} f'(x(z^n),r))\right]\min\left\{\widehat{M}^{\epsilon},1\right\}
\end{aligned} \tag{24}$$

Using (17),(23) and (24), we have

$$\left(\frac{\overline{z}-h}{1-(\overline{z}-h)}\right)^{-\rho_0} \geq \overline{z}^{-\rho_0}\delta E\left[(B(r))^{1-\rho_0}\right](\widehat{M})^{\epsilon}.$$

As h, ϵ are arbitrary (and $\widehat{M}$ is independent of h, ϵ), we have

$$\left(\frac{\overline{z}}{1-\overline{z}}\right)^{-\rho_0} \geq \overline{z}^{-\rho_0}\delta E\left[(B(r))^{1-\rho_0}\right]$$

so that

$$(1-\overline{z})^{\rho_0} \geq \delta E\left[(B(r))^{1-\rho_0}\right] = (s_0)^{\rho_0} \tag{25}$$

This establishes (13) and also implies that $s_0 < 1$. The proof is complete. ■

The next result is a corollary of the previous lemma:

Corollary 1

$$\frac{c(y)}{y} \to 0, \textit{ if } s_0 \geq 1 \tag{26}$$

and

$$\lim_{y\to 0}\sup\frac{c(y)}{y} \leq 1-s_0, \textit{ if } s_0 < 1. \tag{27}$$

Proof. If if $s_0 \geq 1$ and $\lim_{y\to 0}\sup\frac{c(y)}{y} > 0$, we have a contradiction to Lemma 4 using (26). On the other hand, if $s_0 < 1$,(27) obviously holds if $\lim_{y\to 0}\sup\frac{c(y)}{y} = 0$; if $\lim_{y\to 0}\sup\frac{c(y)}{y} > 0$, (27) follows from Lemma 4. ■

The next lemma indicates that the upper bound on the limiting optimal propensity to consume at zero outlined in inequality (27) of Corollary 1 is the exact limit as long as the optimal consumption propensity is bounded away from zero.

Lemma 5 *Suppose that*

$$\liminf_{y \to 0} \frac{c(y)}{y} > 0. \tag{28}$$

then

$$\lim_{y \to 0} \frac{c(y)}{y} = 1 - s_0 \tag{29}$$

Proof. Let

$$\underline{z} = \liminf_{y \to 0} \frac{c(y)}{y}, \overline{z} = \limsup_{y \to 0} \frac{c(y)}{y}$$

Using Lemma 3 and (28)

$$0 < \underline{z} \leq \overline{z} < 1. \tag{30}$$

Further, from Lemma 4, $s_0 < 1$ and $\overline{z} \leq 1 - s_0$. We will show that if (28) holds, then

$$\underline{z} \geq 1 - s_0 \tag{31}$$

so that (using (30)), $\underline{z} = \overline{z} = 1 - s_0$ and (29) holds. Fix $\widehat{h} \in (0, \min\{\underline{z}, 1 - \overline{z}\})$ and $\widehat{\lambda} \in (0, 1)$. Choose any ϵ, h such that $0 < \epsilon < \rho_0, h \in (0, \widehat{h})$.There exists $\widehat{y} > 0$ such that

$$\rho_0 - \epsilon \leq \rho(c) \leq \rho_0 + \epsilon \text{ for all } c \in (0, \overline{f}(\widehat{y})). \tag{32}$$

and

$$f'(y, r) \geq \lambda \underline{B} \text{ for all } y \in (0, \widehat{y}) \tag{33}$$

By definition of $\underline{z}$ and without loss of generality, there exists a sequence $\{y^n\}_{n=1}^{\infty}$, such that $y^n \in (0, \widehat{y})$ for all n, $y^n \to 0$ as $n \to \infty$, $\{\frac{c(y^n)}{y^n}\}$ is convergent and for all n

$$\underline{z} + h \geq \frac{c(y^n)}{y^n} \geq \underline{z} - h \text{ for all } n \tag{34}$$

which also implies

$$1 - \underline{z} - h \leq \frac{x(y^n)}{y^n} \leq 1 - \underline{z} + h \text{ for all } n \tag{35}$$

Note that $\widehat{h} \in (0, \min\{\underline{z}, 1 - \overline{z}\})$ and $h \in (0, \widehat{h})$ implies that the right hand expression of the second inequality in (34) and the left hand expression of the first inequality in

(35) are strictly positive. Then, for $\rho > 0$

$$\left(\frac{\underline{z}+h}{1-(\underline{z}+h)}\right)^{-\rho} \leq \left(\frac{c(y^n)}{y^n}\right)^{-\rho}\left(\frac{x(y^n)}{y^n}\right)^{\rho} \leq \left(\frac{\underline{z}-h}{1-(\underline{z}-h)}\right)^{-\rho} \tag{36}$$

Let

$$M = \frac{(\underline{z}+\widehat{h})}{(1-(\underline{z}+\widehat{h}))(\underline{z}-\widehat{h})\lambda\underline{B}} \tag{37}$$

$\widehat{h} \in (0, \min\{\underline{z}, 1-\overline{z}\})$ and (30) implies that $0 < M < \infty$. Further,

$$\frac{(\underline{z}+h)}{(\underline{z}-h)(1-(\underline{z}+h))\lambda\underline{B}} \leq M \text{ for all } h \in (0, \widehat{h}). \tag{38}$$

From the Ramsey-Euler equation (7) we have for each n:

$$\begin{aligned}
\frac{1}{\delta} &= E\left[\frac{u'(c(f(x(y^n),r)))}{u'(c(y^n))}f'(x(y^n),r)\right] \\
&\leq E\left[\left(\frac{c(f(x(y^n),r))}{c(y^n)}\right)^{-(\rho_0-\epsilon)} f'(x(y^n),r)I_{[f(x(y^n),r)\geq y^n]}\right] \\
&\quad + E\left[\left(\frac{c(f(x(y^n),r))}{c(y^n)}\right)^{-(\rho_0+\epsilon)} f'(x(y^n),r)I_{[f(x(y^n),r)< y^n]}\right]
\end{aligned} \tag{39}$$

where the inequality follows from Lemma 2 and (32). Observe that if $f(x(y^n),r) \geq y^n$

$$\begin{aligned}
\left(\frac{c(f(x(y^n),r))}{c(y^n)}\right)^{-(\rho_0-\epsilon)} &\leq \left(\frac{c(f(x(y^n),r))}{c(y^n)}\right)^{-\rho_0} \\
&= \left(\frac{x(y^n)}{c(y^n)}\right)^{-\rho_0}\left(\frac{c(f(x(y^n),r))}{f(x(y^n),r)}\frac{f(x(y^n),r)}{x(y^n)}\right)^{-\rho_0}
\end{aligned}$$

so that

$$\begin{aligned}
& E\left[\left(\frac{c(f(x(y^n),r))}{c(y^n)}\right)^{-(\rho_0-\epsilon)} f'(x(y^n),r)I_{[f(x(y^n),r)\geq y^n]}\right] \\
\leq\ & \left(\frac{x(y^n)}{c(y^n)}\right)^{-\rho_0} E\left[\left(\frac{c(f(x(y^n),r))}{f(x(y^n),r)}\right)^{-\rho_0}\left(\frac{f(x(y^n),r)}{x(y^n)}\right)^{1-\rho_0} I_{[f(x(y^n),r)\geq y^n]}\right]
\end{aligned} \tag{40}$$

On the other hand, if $f(x(y^n),r) < y^n$

$$
\begin{aligned}
& \left(\frac{c(f(x(y^n),r))}{c(y^n)}\right)^{-(\rho_0+\epsilon)} \\
= & \left(\frac{x(y^n)}{c(y^n)}\right)^{-(\rho_0+\epsilon)} \left(\frac{c(f(x(y^n),r))}{f(x(y^n),r)}\right)^{-(\rho_0+\epsilon)} \left(\frac{f(x(y^n),r)}{x(y^n)}\right)^{-(\rho_0+\epsilon)} \\
\leq & \left(\frac{x(y^n)}{c(y^n)}\right)^{-\rho_0} \left(\frac{1-(\underline{z}+h)}{\underline{z}+h}\right)^{-\epsilon} \left(\frac{c(f(x(y^n),r))}{f(x(y^n),r)}\right)^{-\rho_0} (\underline{z}-h)^{-\epsilon} \left(\frac{f(x(y^n),r)}{x(y^n)}\right)^{-\rho_0} (\lambda\underline{B})^{-\epsilon} \\
= & \left(\frac{x(y^n)}{c(y^n)}\right)^{-\rho_0} \left(\frac{c(f(x(y^n),r))}{f(x(y^n),r)}\right)^{-\rho_0} \left(\frac{f(x(y^n),r)}{x(y^n)}\right)^{-\rho_0} \left(\frac{(1-\underline{z}-h)(\underline{z}-h)}{\underline{z}+h}\lambda\underline{B}\right)^{-\epsilon} \\
\leq & \left(\frac{x(y^n)}{c(y^n)}\right)^{-\rho_0} \left(\frac{c(f(x(y^n),r))}{f(x(y^n),r)}\right)^{-\rho_0} \left(\frac{f(x(y^n),r)}{x(y^n)}\right)^{-\rho_0} M^{\epsilon}
\end{aligned}
$$

where the first inequality follows from (16),(34) and (35) and the second inequality uses (37). Thus,

$$
\begin{aligned}
& E\left[\left(\frac{c(f(x(y^n),r))}{c(y^n)}\right)^{-(\rho_0+\epsilon)} f'(x(y^n),r) I_{[f(x(y^n),r)<y^n]}\right] \\
\leq & \left(\frac{x(y^n)}{c(y^n)}\right)^{-\rho_0} E\left[\left(\frac{c(f(x(y^n),r))}{f(x(y^n),r)}\right)^{-\rho_0} \left(\frac{f(x(y^n),r)}{x(y^n)}\right)^{1-\rho_0} I_{[f(x(y^n),r)<y^n]}\right] M^{\epsilon}
\end{aligned}
$$

and combining this with (39) and (40), we have

$$
\frac{1}{\delta} \leq \left(\frac{x(y^n)}{c(y^n)}\right)^{-\rho_0} E\left[\left(\frac{c(f(x(y^n),r))}{f(x(y^n),r)}\right)^{-\rho_0} \left(\frac{f(x(y^n),r)}{x(y^n)}\right)^{1-\rho_0}\right] \max\{M^{\epsilon},1\}
$$

so that using (36)

$$
\left(\frac{\underline{z}+h}{1-(\underline{z}+h)}\right)^{-\rho_0} \leq \left(\frac{x(y^n)}{c(y^n)}\right)^{-\rho_0} \leq \delta E\left[\left(\frac{c(f(x(y^n),r))}{f(x(y^n),r)}\right)^{-\rho_0} \left(\frac{f(x(y^n),r)}{x(y^n)}\right)^{1-\rho_0}\right] \max\{M^{\epsilon},1\} \tag{41}
$$

For each $r \in A$,

$$
\limsup_{n\to\infty} \left[\left(\frac{c(f(x(y^n),r))}{f(x(y^n),r)}\right)^{-\rho_0} \left(\frac{f(x(y^n),r)}{x(y^n)}\right)^{1-\rho_0}\right] \leq \underline{z}^{-\rho_0}(B(r))^{1-\rho_0} \tag{42}
$$

Note that $\{\frac{c(f(x(y^n),r))}{f(x(y^n),r)}\}$ is bounded away from zero so that

$$\left(\frac{c(f(x(y^n),r))}{f(x(y^n),r)}\right)^{-\rho_0}\left(\frac{f(x(y^n),r)}{x(y^n)}\right)^{1-\rho_0}$$

is uniformly bounded above by an integrable function; taking the limsup as $n\to\infty$ on both sides of (41):

$$\begin{aligned}&\left(\frac{\underline{z}+h}{1-(\underline{z}+h)}\right)^{-\rho_0}\\ \leq\ &\delta E\left[\limsup_{n\to\infty}\left(\frac{c(f(x(y^n),r))}{f(x(y^n),r)}\right)^{-\rho_0}\left(\frac{f(x(y^n),r)}{x(y^n)}\right)^{1-\rho_0}\right]\max\{M^{\epsilon},1\}\end{aligned}$$

(see, for instance, Royden, 1988: Problem 12, Chapter 4), and using (42) we have

$$\left(\frac{\underline{z}+h}{1-(\underline{z}+h)}\right)^{-\rho_0}\leq \underline{z}^{-\rho_0}\delta E\left[(B(r))^{1-\rho_0}\right]\max\{M^{\epsilon},1\}.$$

As this inequality is shown to hold for all $h\in(0,\widehat{h})$ and $\epsilon\in(0,\rho_0)$ and as M is independent of ϵ,h, we have

$$\left(\frac{\underline{z}}{1-\underline{z}}\right)^{-\rho_0}\leq \underline{z}^{-\rho_0}\delta E\left[(B(r))^{1-\rho_0}\right]$$

so that

$$(1-\underline{z})^{\rho_0}\leq\delta E\left[(B(r))^{1-\rho_0}\right]=(s_0)^{\rho_0}$$

which yields (31). This completes the proof the lemma. ■

Proof of Proposition 1:

Part (i) of the proposition follows from Corollary 1; in particular, (27) implies that if $s_0<1,\lim_{y\to0}\inf\frac{x(y)}{y}\geq s_0=\theta$. On the other hand, $s_0\geq1$ implies $\theta=1$ and using (26), we have $\frac{x(y)}{y}\to1=\theta$ as $y\to0$. This establishes (10). Part (ii) of the proposition follows directly from Lemma 5.

4 Regeneration of Capital

In this section, we outline our main result on regeneration of capital near zero. In particular, we use the bound on the optimal propensity to invest near zero characterized in Proposition 1 to derive an explicit condition on the economic fundamentals under which the optimal policy is such that capital regenerates near zero and consumption is strictly positive in the long run with probability one.

We begin by imposing the following assumption on the production function:

(**T.3**)

$$\frac{f(x,r)}{x} \to B(r) \ \text{ as } x \to 0 \text{ uniformly in } r \text{ on } A.$$

Note that (**T.3**) is satisfied if the random shock is multiplicative (for instance, $f(x,r) = rh(x)$) and A, the support of the distribution of random shocks, is a bounded subset of $\mathbb{R}_{++}$.

Our condition for almost sure regeneration of capital near zero is as follows:

Condition R:

$$E[\ln(\theta B(r))] > 0 \tag{R}$$

Note that θ depends on the discount factor, the degree of risk aversion, the distribution of random shocks and the marginal productivity of capital and therefore these factors also determine whether or not Condition **R** holds. In particular, if $\theta < 1$ so that $\theta = s_0 = (\delta E((B(r))^{1-\rho_0})^{1/\rho_0}$, Condition **R** is equivalent to:

$$(1/\rho_0)\left[\ln\delta + \ln E\left((B(r))^{1-\rho_0}\right)\right] + E\ln(B(r)) > 0$$

which can be written as

$$E[\ln(\delta B(r))] + \left[\left\{\ln E\left((B(r))^{1-\rho_0}\right)\right\} - \left\{E\ln(B(r)^{1-\rho_0})\right\}\right] > 0 \tag{43}$$

Using Jensen's inequality, the second term in square brackets on the left hand side of inequality (43) reflects the interaction between risk aversion (near zero) and riskiness of the productivity shock; it is always non-negative and it is strictly positive if$\rho_0 \neq 1$**.**

We will show that Condition **R** ensures the following. First, optimal outputs are (loosely speaking) uniformly bounded away "in probability" from zero. Second, even if optimal output, and therefore investment and consumption, reach arbitrarily small levels from time to time, they will almost surely rebound instead of converging to zero; eventual extinction is never optimal.

Proposition 2 *Assume (**T.3**) and Condition **R**. Then the following hold for all initial stocks $y \in \mathbb{R}_{++}$:*

(i) For any $\xi > 0$, there exists $\widehat{\alpha}(y) > 0$ such that

$$\Pr\{y_t(y) < \widehat{\alpha}(y)\} < \xi \text{ for all } t \in \mathbb{N} \tag{44}$$

(ii)

$$\Pr\{\lim_{t\to\infty} \sup y_t(y) > 0\} = 1 \tag{45}$$

so that $\lim_{t\to\infty} \sup c_t(y) > 0$ and $\lim_{t\to\infty} \sup x_t(y) > 0$ with probability one i.e., under the optimal policy, capital stocks and consumption levels remain strictly positive in the long run with probability one.

(iii) Extinction of capital occurs with zero probability i.e.,

$$\Pr\{x_t(y) \to 0\} = 0.$$

Proof. (i) Note that $0 < \theta \le 1$. Condition **R** implies that $E[\ln B(r)] > 0$. Also observe that as $x(y) > 0$ for all $y > 0$,using (4) we have $\underline{H}(y) = \underline{f}(x(y)) > 0$ for all $y > 0$. We begin by showing that the following holds:

$$\exists\, \widetilde{\sigma} > 0 \text{ such that for every } y > 0,\ M(y) = \sup_t E\left\{\left(\frac{1}{y_t(y)}\right)^{\sigma(y)}\right\} < \infty. \tag{46}$$

From Hardy et al (1952, pp. 152-7) we have

$$\lim_{\sigma\downarrow 0} \ln\left[E\left(\frac{1}{\theta B(r)}\right)^{\sigma}\right]^{\frac{1}{\sigma}} = E\ln\left(\frac{1}{\theta B(r)}\right).$$

Using Condition **R**, $E\ln\left(\frac{1}{\theta B(r)}\right) = -E\ln\theta B(r) < 0$ and so there exists $\widetilde{\sigma} > 0$ such that

$$\ln\left[E\left\{\left(\frac{1}{\theta B(r)}\right)^{\widetilde{\sigma}}\right\}\right]^{\frac{1}{\widetilde{\sigma}}} < 0$$

i.e.,

$$E\left(\frac{1}{\theta B(r)}\right)^{\widetilde{\sigma}} < 1. \tag{47}$$

We now show that the sequence $\left\{E\left(\frac{1}{y_t(y)}\right)^{\tilde{\sigma}}\right\}_{t=0}^{\infty}$ is bounded above. Note that $y_t(y)$ is bounded below and above by $\underline{H}^t(y) > 0$ and $\overline{f}^t(y) < \infty$ so that $0 < E\left(\frac{1}{y_t(y)}\right)^{\tilde{\sigma}} < \infty$ for every t. Using (47), there exists $\epsilon > 0$ small enough so that

$$\lambda = E\left(\frac{1}{(1-\epsilon)B(r)\theta}\right)^{\tilde{\sigma}} \in (0,1). \tag{48}$$

Using assumption (**T.3**) and (10), there exists $a > 0$ such that for all $z \in (0, a)$, $r \in A$,

$$\frac{H(z,r)}{z} = \frac{f(x(z),r)}{x(z)}\frac{x(z)}{z} \geq (1-\epsilon)B(r)\theta$$

Let

$$m = \left(\frac{1}{\underline{H}(a)}\right)^{\tilde{\sigma}}$$

Then, $m < \infty$. Note λ, m do not depend on t or the initial stock y. Then,

$$\begin{aligned}
\left(\frac{1}{y_{t+1}(y)}\right)^{\tilde{\sigma}} &= \left(\frac{1}{H(y_t(y), r_{t+1})}\right)^{\tilde{\sigma}} \\
&= \left(\frac{1}{H(y_t(y), r_{t+1})}\right)^{\tilde{\sigma}} I_{\{y_t(y)<a\}} + \left(\frac{1}{H(y_t(y), r_{t+1})}\right)^{\tilde{\sigma}} I_{\{y_t(y)\geq a\}} \\
&\leq \left(\frac{1}{(1-\epsilon)B(r_{t+1})\theta y_t(y)}\right)^{\tilde{\sigma}} I_{\{y_t(y)<a\}} + \left(\frac{1}{\underline{H}(a)}\right)^{\tilde{\sigma}} I_{\{y_t(y)\geq a\}} \\
&\leq \left(\frac{1}{(1-\epsilon)B(r_{t+1})\theta y_t(y)}\right)^{\tilde{\sigma}} + m
\end{aligned}$$

so that taking expectation (with respect to information in time 0):

$$\begin{aligned}
E\left\{\left(\frac{1}{y_{t+1}(y)}\right)^{\tilde{\sigma}}\right\} &\leq E\left\{\left(\frac{1}{(1-\epsilon)B(r_{t+1})\theta}\right)^{\tilde{\sigma}}\right\} E\left\{\left(\frac{1}{y_t(y)}\right)^{\tilde{\sigma}}\right\} + m, \\
&\quad \text{as } y_t(y) \text{ and } r_{t+1} \text{ are independent} \\
&= \lambda E\left[\left(\frac{1}{y_t(y)}\right)^{\tilde{\sigma}}\right] + m, \text{ using (48)}
\end{aligned} \tag{49}$$

From (49) it follows that the sequence $\left\{E\left(\frac{1}{y_t(y)}\right)^{\tilde{\sigma}}\right\}_{t=0}^{\infty}$ is Cauchy and hence, convergent and bounded. Thus, (46) holds. Note that $M(y) > 0$. Now, fix any $y > 0$ and choose any $\xi > 0$. Choose $\widehat{\alpha}(y) \in (0, y)$ such that:

$$\widehat{\alpha}(y) \leq \left(\frac{\xi}{M(y)}\right)^{\frac{1}{\widetilde{\sigma}}}. \tag{50}$$

We will show that for all $t \in \mathbb{N}$,

$$P\{y_t(y) < \widehat{\alpha}(y)\} \leq \xi \tag{51}$$

so that the lemma holds. To see that (51) holds for all t,suppose to the contrary that there is some t for which (51) does not hold i.e.,

$$P\{y_t(y) < \widehat{\alpha}(y)\} > \xi. \tag{52}$$

Then,

$$\begin{aligned} E\left\{\left(\frac{1}{y_t(y)}\right)^{\widetilde{\sigma}}\right\} &= E\left[\left(\frac{1}{y_t(y)}\right)^{\widetilde{\sigma}} I_{[y_t(y)<\widehat{\alpha}(y)]} + \left(\frac{1}{y_t(y)}\right)^{\widetilde{\sigma}} I_{[y_t(y)\geq\widehat{\alpha}(y)]}\right] \\ \geq\ & E\left[\left(\frac{1}{y_t(y)}\right)^{\widetilde{\sigma}} I_{[y_t(y)<\widehat{\alpha}(y)]}\right] \geq \left(\frac{1}{\widehat{\alpha}(y)}\right)^{\widetilde{\sigma}} P\{y_t(y) < \widehat{\alpha}(y)\} \\ >\ & \left(\frac{1}{\widehat{\alpha}(y)}\right)^{\widetilde{\sigma}} \xi \text{ , using (52),} \\ \geq\ & M(y), \text{ using (50),} \end{aligned}$$

which contradicts (46). Thus (51) holds for all t. This establishes (44).

(ii) As $c(\widetilde{y}) > 0$ and $x(\widetilde{y}) > 0$ for all $\widetilde{y} > 0$, it is sufficient to show that (45) holds. To see (45), choose any $\xi > 0$ and let $\widehat{\alpha}(y) > 0$ be as described in the proof of part (i) of the proposition. Observe that

$$\{\omega \in \Omega : \lim_{t\to\infty} y_t(y) = 0\} \subset \cup_{T=0}^{\infty}\{\omega \in \Omega : y_t(y) < \widehat{\alpha}(y) \text{ for all } t \geq T\}$$

and as the as the sets $\{\omega \in \Omega : y_t(y) < \widehat{\alpha}(y) \text{ for all } t \geq T\}$ are nested and expanding in T,

$$\begin{aligned} \Pr\{\lim_{t\to\infty} y_t(y) &= 0\} \leq \lim_{T\to\infty} \Pr\{y_t(y) < \widehat{\alpha}(y) \text{ for all } t \geq T\} \\ &\leq \lim_{T\to\infty} \sup \Pr\{y_T(y) < \widehat{\alpha}(y)\} < \xi, \text{ using (44).} \end{aligned}$$

As ξ is arbitrary, $\Pr\{\lim_{t\to\infty} y_t(y) = 0\} = 0$ i.e., $\Pr\{\lim_{t\to\infty} \sup y_t(y) > 0\} = 1$.

(iii) Follows immediately from (ii). ■

5 Tightness of Condition R

In this section, we illustrate the tightness of our general condition (Condition **R)** for regeneration of capital near zero. We show that for a class of utility and production functions that are widely used in the literature, a strict violation of Condition **R** implies that all optimal paths converge to zero i.e., eventual extinction occurs with probability one from all positive initial stocks. This result is stated as Proposition 4 in subsection 5.2.

The restricted family of utility and production functions that we consider in this section are as follows. First, we confine attention to utility functions that satisfy constant relative risk aversion i.e., we assume that:

(U.4)

$$\begin{aligned} u(c) &= \frac{c^{1-\rho_0}}{1-\rho_0}, \rho_0 > 0, \rho_0 \neq 1 \\ &= \ln c \text{ (corresponding to } \rho_0 = 1). \end{aligned}$$

Second, we impose following joint restriction on the set of admissible production and utility functions:

Condition B: For all $r \in A, x \in \mathbb{R}_{++}$,

$$\frac{f'(x,r)x^{\rho_0}}{(f(x,r))^{\rho_0}} \leq (B(r))^{1-\rho_0} \qquad (B)$$

Note that

$$\lim_{x\to 0} \left\{ \frac{f'(x,r)x^{\rho_0}}{(f(x,r))^{\rho_0}} \right\} = (B(r))^{1-\rho_0}$$

so that Condition **B** essentially requires that the function $\frac{f'(x,r)x^{\rho_0}}{(f(x,r))^{\rho_0}}$ is "maximized at zero". The required inequality in (B) can be rewritten as:

$$\left(\frac{f'(x,r)}{f'(0,r)}\right)^{1-\rho_0} \left[\frac{f'(x,r)x}{f(x,r)}\right]^{\rho_0} \leq 1, \forall x \in \mathbb{R}_{++}, r \in A$$

which always holds if $\rho_0 \in (0,1]$ but can also hold if $\rho_0 > 1$.[13]

Example 1 *Consider the family of production functions:*

$$\begin{aligned} f(x,r) &= 0, \text{ if } x = 0 \\ &= r(x^{1-\eta}+\beta)^{\frac{1}{1-\eta}}, \beta \geq 0, \eta > 1 \end{aligned}$$

*Note that $f(x,r)$ satisfies assumptions **(T.1)**-**(T.3)**. If $\beta > 0$, f exhibits bounded growth and Condition* **B** *is satisfied as long as $\rho_0 \leq \eta$.*[14] *If $\beta = 0$, f is a linear production function and Condition* **B** *holds for all $\rho_0 > 0$.*

5.1 Upper bound on the Optimal Investment Function

In Section 3, we have shown that θ is always a lower bound on the optimal propensity to invest near zero. We now show that for the family of utility and production functions outlined above, θ is also an upper bound on the optimal propensity to invest; further, this holds not just near zero but at any level of output i.e., we have an upper bound on the optimal investment function. This is an important step towards showing that optimal paths may converge to zero when Condition **R** does not hold. It can be a useful result for other purposes.

Proposition 3 *Assume **(U.4)** and Condition B. Then, the optimal propensity to invest is bounded above by θ on $\mathbb{R}_{++}$*

$$\frac{x(y)}{y} \leq \theta \text{ for all } y \in \mathbb{R}_{++} \tag{53}$$

[13]If f is twice differentiable and $\eta_1(x,r), \eta_2(x,r)$ are the first and second elasticities of the production function defined by

$$\eta_1(x,r) = \frac{f'(x,r)x}{f(x,r)}, \eta_2(x,r) = -\frac{f''(x,r)x}{f'(x,r)},$$

then Condition B holds if

$$\rho_0 \leq \frac{\eta_2(x,r)}{1-\eta_1(x,r)}, \forall x \in \mathbb{R}_{++}, r \in A.$$

[14]If $\rho_0 = \eta$, the optimal policy function is linear and the optimal propensity to invest is θ; see, among others, Benhabib and Rustichini (1994), Mitra and Sorger (2014).

5.1.1 Proof of Proposition 3

If $\theta = 1$, Proposition 3 is trivial. So, we focus on $\theta < 1$ in which case $\theta = s_0$. The proof first shows that θ is an upper bound on the optimal propensity to invest in the finite horizon version of the dynamic optimization problem in Section 2 and then uses policy convergence to extend this to the optimal policy function for the infinite horizon problem.

Consider finite horizon version of the stationary stochastic dynamic optimization problem outlined in section 2. In particular, for $T \in \mathbb{N}$, and given initial stock $y \geq 0$, the agent maximizes:

$$E\left[\sum_{t=0}^{T} \delta^t u(\widetilde{c}_t)\right]$$

over feasible a stochastic processes $\{\widetilde{y}_t, \widetilde{c}_t, \widetilde{x}_t\}_{t=0}^{T}$ where $\widetilde{y}_0 = y$, where $\{\widetilde{c}_t, \widetilde{x}_t\}$ are $\mathcal{F}_t$ adapted where $\mathcal{F}_t$ is the (sub) σ-field generated by partial history from periods 0 through t and:

$$\begin{aligned}&(i)\ \widetilde{c}_t \geq 0, \widetilde{x}_t \geq 0 \ \text{ for } t = 0, 1, ...T\\ &(ii)\ \widetilde{c}_t + \widetilde{x}_t \leq \widetilde{y}_t,\ \widetilde{y}_{t+1} = f(\widetilde{x}_t, r_{t+1}.) \text{ for } t = 0, 1, ...T\end{aligned}$$

Note that here is no terminal stock requirement in period T. Standard arguments can be used to establish that there exists a unique optimal decision rule in each period t and it depends only on the number of periods left till the end of the time horizon.[15]

Lemma 6 *Consider the $T-$period finite horizon problem. There exist (unique) optimal consumption and investment functions denoted by $c^\tau(y)$ and $x^\tau(y)$ that depend only on τ, the number of periods remaining till the end of the time horizon; in any period $t = T - \tau$, it is optimal to consume $c^\tau(y)$ and invest $x^\tau(y)$ if current output is y. $c^\tau(y) > 0, x^\tau(y) > 0$ for all $\tau \in \mathbb{N}$ and $y > 0$. For every $\tau \in \mathbb{N}_+$, $c^\tau(y)$ and $x^\tau(y)$ are continuous and strictly increasing on $\mathbb{R}_+$. The following stochastic Ramsey-Euler equation folds for all $\tau \in \mathbb{N}$ and $y > 0$:*

$$u'(c^{\tau+1}(y)) = \delta E[u'(c^\tau(f(x^{\tau+1}(y), r)))f'(x^{\tau+1}(y), r)] \tag{54}$$

Proof. Using induction on τ and fairly standard arguments as in the infinite horizon case. ■

[15] See, among others, Majumdar and Zilcha (1987)

The next lemma establishes a uniform *upper* bound on the optimal propensity to invest in finite horizon problems.

Lemma 7 *Assume (**U.4**) and Condition B. Further, suppose that $\theta < 1$. Then, the following hold:*

(i) For every $\tau \in \mathbb{N}$ and $y > 0$.

$$\frac{x^{\tau}(y)}{y} \leq \theta \text{ for all } y > 0 \tag{55}$$

(ii) The finite horizon optimal investment functions converge point-wise to the optimal investment function for the infinite horizon problem i.e.,

$$\lim_{\tau \to \infty} x^{\tau}(y) = x(y) \text{ for all } y \geq 0$$

Proof. (i) Note that as $\theta < 1$

$$\theta = s_0 = \left[E((B(r))^{1-\rho_0})\right]^{1/\rho_0} < 1.$$

As $x^0(y) = 0$ for all $y > 0$, (55) holds for $\tau = 0$. Suppose (55) holds for $\tau = t \in \mathbb{N}$. For every $y > 0$, we have from (54) that:

$$(c^{t+1}(y))^{-\rho_0} = \delta E[(c^t(f(x^{t+1}(y), r)))^{-\rho_0} f'(x^{t+1}(y), r)]$$

and as (55) holds for $\tau = t, c^t(y) \geq (1 - \theta)y$ for all $y > 0$, and therefore,

$$\begin{aligned}(c^{t+1}(y))^{-\rho_0} &= \delta E[(c^t(f(x^{t+1}(y), r)))^{-\rho_0} f'(x^{t+1}(y), r)] \\ &\leq \delta E[((1-\theta)(f(x^{t+1}(y), r)))^{-\rho_0} f'(x^{t+1}(y), r)]\end{aligned}$$

which implies that:

$$\begin{aligned}\left(\frac{c^{t+1}(y)}{y}\right)^{-\rho_0} &\leq \delta(1-\theta)^{-\rho_0} E\left[\left(\frac{f(x^{t+1}(y), r)}{x^{t+1}(y), r}\right)^{-\rho_0} f'(x^{t+1}(y), r)\right]\left[\frac{x^{t+1}(y)}{y}\right]^{-\rho_0} \\ &\leq \delta(1-\theta)^{-\rho_0} E\left[(B(r))^{1-\rho_0}\right]\left[\frac{x^{t+1}(y)}{y}\right]^{-\rho_0}, \text{ using Condition } B \\ &= (1-\theta)^{-\rho_0}\theta^{\rho_0}\left[\frac{x^{t+1}(y)}{y}\right]^{-\rho_0}\end{aligned}$$

so that

$$\left(\frac{x^{t+1}(y)}{y}\right)\left(1-\frac{x^{t+1}(y)}{y}\right)^{-1} \leq \frac{\theta}{1-\theta}$$

which yields:

$$\frac{x^{t+1}(y)}{y} \leq \theta.$$

Thus, (55) holds for all $\tau \in \mathbb{N}$.

(ii) The policy convergence follows from sufficient conditions in Schäl (1975). In particular, one can check that both conditions (GA) and (C) in Section 2 of that paper are satisfied.[16] ■

Proposition 3 follows immediately from parts (i) and (ii) of the above lemma.

5.2 Global Extinction

We are now ready state the main result of this section:

Proposition 4 *Assume* (**U.4**) *and Condition B. If*

$$E[\ln(\theta B(r))] < 0 \tag{56}$$

then almost sure (eventual) extinction is optimal from all initial stocks i.e., $\{y_t(y), c_t(y), x_t(y)\} \to 0$ *almost surely for all* $y \in \mathbb{R}_+$.

Proof. First, consider the case where $E[\ln(B(r))] < 0$. We will show that in this case, every *feasible* stochastic process must converge to zero with probability one. This is similar to the result reported in Kamihigashi (2006). Given any initial stock $y > 0$, any feasible stochastic process $\{\widetilde{y}_t(y), \widetilde{c}_t(y), \widetilde{x}_t(y)\}$ satisfies $\widetilde{y}_t(y) \leq y_t^M(y)$ where $\{y_t^M(y)\}$ is the stochastic process defined by $y_0^M = y, y_{t+1}^M = f(y_t^M, r_{t+1})$ for all $t \geq 0$; as $f(y_t^M, r) \leq B(r)y_t^M$ for all $r \in A$, it is easy to check that for all $t \geq 1$:

$$\ln \widetilde{y}_t(y) \leq \ln y_t^M \leq \ln y + \sum_{i=1}^{t} \ln B(r_t) = \ln y + t\left[\frac{1}{t}\sum_{i=1}^{t} \ln B(r_t)\right] \tag{57}$$

and as r_t's are i.i.d., the strong law of large numbers implies $\frac{1}{t}\sum_{i=1}^{t} \ln B(r_t) \to E[\ln(B(r))]$ with probability one as $t \to \infty$; $E[\ln(B(r))] < 0$ then implies that as

[16] If $\rho_0 > 1$, we are in the "negative" case in Schäl (1975) and condition (C) specified in that paper always holds (see discussion in Section 2 of that paper). If $\rho_0 \leq 1$, assumption **(D.1)** implies that condition (C) in Schäl (1975) holds.

$t \to \infty$, the right hand side of (57) converges to$-\infty$ and $\widetilde{y}_t(y) \to 0$ with probability one.

Next, we consider the case where $E[\ln(B(r))] \geq 0$. Condition (56) of the proposition then implies that $\theta < 1$ i.e., $\theta = s_0 < 1$. Consider the stochastic process $\{y_t(y), c_t(y), x_t(y)\}$ generated by the optimal policy from initial stock $y \geq 0$. This is trivial if $y = 0$. So, consider $y > 0$. Using Proposition 3 we have

$$H(y, r) = f(x(y), r) = \frac{f(x(y), r)}{x(y)} x(y) \leq B(r)x(y) \leq B(r)\theta y$$

so that for all $t \geq 1$

$$y_t(y) = H(y_{t-1}(y), r_t) \leq B(r_t)\theta y_{t-1}(y)$$

which can be used to show that

$$\ln y_t(y) \leq \ln y + \sum_{i=1}^{t} \ln \theta B(r_t) = \ln y + t\left[\frac{1}{t}\sum_{i=1}^{t} \ln \theta B(r_t)\right]$$

and using similar arguments as above, we have that (56) implies that as $t \to \infty$, $y_t(y) \to 0$ with probability one. ■

6 Risk Aversion and Regeneration

Our condition for regeneration (Condition **R**) outlined in Proposition 2 allows us to study the effect of change in risk aversion (near zero) on the optimality of regeneration of capital (i.e., avoidance of extinction) and positive long run consumption.

We begin by showing that if Condition **R** is satisfied when ρ_0, the (limiting) Arrow-Pratt relative risk aversion at zero consumption, is equal to 1 (such as in the case of the log utility function), then it is satisfied for all admissible utility functions; in this case, change in risk aversion does not affect the optimality of regeneration. Note that at $\rho_0 = 1$, $\theta = s_0 = \delta$ so that Condition **R** is equivalent to requiring $E[\ln(\delta B(r))] > 0$.

Proposition 5 *Assume* ***(T.3)****. Suppose that*

$$E[\ln(\delta B(r))] > 0 \tag{58}$$

Then regardless of the degree of risk aversion i.e., for all $\rho_0 > 0$, it is optimal to regenerate capital near zero with probability one and in particular, the conclusions of Proposition 2 always hold.

Proof. Note that (58) implies $E[\ln B(r)] > 0$ which implies that Condition **R** holds if $\theta = 1$. If $\theta < 1$, then as mentioned in Section 4, Condition R holds if and only if (43) holds. Using Jensen's inequality,

$$\left[\left\{\ln E\left((B(r))^{1-\rho_0}\right)\right\} - \left\{E\ln(B(r)^{1-\rho_0})\right\}\right] \geq 0$$

so that (58) implies that (43) holds. ■

It is worth noting that (58) is the condition imposed in Kamihigashi (2007) to ensure avoidance of extinction and convergence to a positive steady state; in fact, it is the weakest such condition in the existing literature. Condition **R** is weaker than (58) and they coincide only if $\rho_0 = 1$.

Next, we consider the situation where Condition **R** is *not* satisfied when the risk aversion parameter $\rho_0 = 1$ and in particular, $E[\ln(\delta B(r))] < 0$. Here, change in risk aversion can alter the possibility of regeneration. In particular, we show that in certain situations, regeneration may be optimal when the degree of risk aversion is either low or high, but global extinction may be optimal when risk aversion is at an intermediate level.

Proposition 6 *Assume **(T.3)**. Suppose that*

$$E[\ln(\delta B(r))] < 0 < E[\ln B(r)]$$

Then the following hold:

*(a) If ρ_0 is close to 1, then for utility and production functions that satisfy **(U.4)** and Condition B, eventual extinction with probability one is optimal from all initial stocks.*

(b) There exists $\overline{\rho} > 1$ such that for all $\rho_0 > \overline{\rho}$, almost sure regeneration near zero is optimal and in particular, the conclusions of Proposition 2 hold.

(c) If, further, $\delta EB(r) > 1$, there exists $\underline{\rho} \in (0,1)$ such that for all $\rho_0 \in (0, \underline{\rho})$, almost sure regeneration near zero is optimal and in particular, the conclusions of Proposition 2 hold.

Proof. (a) At $\rho_0 = 1$, $\theta = \delta$ so that $E[\ln(\delta B(r))] < 0$ implies that (56) holds forρ_0 close to 1 (θ being continuous in ρ_0); the result then follows from Proposition 4.

(b) Consider $\rho_0 > 1$. If $\theta = 1$, then $0 < E[\ln B(r)]$ immediately implies Condition **R** for regeneration holds. So, consider $\theta < 1$. Here, $\theta = s_0$ in which case Condition **R** holds if, and only if, (43) holds i.e.,

$$E[\ln(\delta B(r))] + q(\rho_0) > 0 \tag{59}$$

where the function $q(\rho)$ is given by

$$q(\rho) = \left[\left\{\ln E\left((B(r))^{1-\rho}\right)\right\} - \left\{E\ln(B(r)^{1-\rho})\right\}\right] > 0$$

for$\rho > 1$ as the distribution of $B(r)$ is non-degenerate. We will show that $q(\rho) \to +\infty$ as $\rho \to \infty$ so that (59) (and therefore Condition **R)** holds for all ρ_0 large enough. Observe that

$$q(3) = \left[\ln E\left(\frac{1}{B(r)}\right)^2 + 2E\ln B(r)\right] > 0$$

Consider any$\rho > 3$. Then

$$\begin{aligned} q(\rho) &= (\rho-1)\left[\frac{1}{\rho-1}\left\{\ln E\left((B(r))^{1-\rho}\right)\right\} + E\ln(B(r))\right] \\ &= (\rho-1)\left[\ln\left\{E\left(\frac{1}{B(r)}\right)^{\rho-1}\right\}^{\frac{1}{\rho-1}} + E\ln(B(r))\right] \\ &\geq (\rho-1)\left[\frac{1}{2}\ln\left\{E\left(\frac{1}{B(r)}\right)^{2}\right\} + E\ln(B(r))\right] \\ &= \frac{(\rho-1)}{2}q(3) \to +\infty \text{ as } \rho \to \infty. \end{aligned}$$

where the inequality in the third line follows from Liapounov's Inequality[17]. This completes the proof of part (b).

(c) As $s_0 \to \delta EB(r)$ when $\rho_0 \to 0$, $\delta EB(r) > 1$ implies that $\theta = 1$ for all ρ_0 close enough to 0; $0 < E[\ln B(r)]$ then implies that Condition **R** holds for ρ_0 small enough. ■

[17] See Chung (1974, pp. 47): $E(|X|^k)^{\frac{1}{k}} \leq E(|X|^m)^{\frac{1}{m}}$ where $1 < k < m < \infty$.

7 Implications for Positive Steady State

Our condition for regeneration (Condition **R**) ensures that with probability one, optimal paths do not converge to zero and optimal outputs are strictly positive in the long run with probability one. This indicates that if the stochastic process of optimal output from any strictly positive initial stock converges in distribution to an invariant distribution, then the support of the limit distribution is in $\mathbb{R}_{++}$ (it cannot assign strictly positive probability to zero); such an invariant distribution would be the stochastic analogue of a non-zero steady state in the deterministic growth model and we can refer to it as a positive stochastic steady state. Under the convex structure of our model, one may expect the steady state to be globally stable.

As mentioned in the introduction, the existing literature has identified conditions that ensure a globally stable positive stochastic steady state. Our condition for regeneration and Proposition 2 can be used to weaken these conditions. In particular, if the production function satisfies bounded growth so that feasible outputs are almost surely bounded above by some maximum sustainable output $K < \infty$, Proposition 2(i) would imply that under Condition **R,** the stochastic kernel of the optimal output process is "bounded in probability" on $(0, K]$; this, in turn, allows us to use some recent results on global stability of monotone stochastic process by Kamihigashi and Stachurski (2014) to establish the existence of a globally stable positive stochastic steady state. These arguments are formally outlined in the Appendix.

One implication of our previous analysis is the important role of consumption utility and risk aversion in convergence to a positive steady state. In particular, the analysis in Section 6 indicates that for a bounded growth technology, global convergence to a positive stochastic steady state is generally ensured if the degree of relative risk aversion near zero is either sufficiently high or sufficiently low, but optimal paths may converge to zero almost surely for intermediate degrees of risk aversion.

Appendix

Bounded Growth Technology: Globally Stable Positive Steady State

In this appendix, we consider the case where the production technology exhibits bounded growth i.e., from any initial stock, feasible consumption, investment and output processes are uniformly bounded above. In particular, the production function satisfies:

(T.4)

$$\lim_{x\to\infty} \sup \left[\frac{\overline{f}(x)}{x}\right] < 1$$

Define the maximum sustainable stock $K \geq 0$ as

$$K = \sup\left\{x \geq 0 : \overline{f}(x) \geq x\right\} \tag{60}$$

Assumption (**T.4**) ensures that $K < \infty$. For technical convenience we also assume that:

(T.5) $\underline{f}$ and $\overline{f}$ are continuous and strictly increasing on $\mathbb{R}_+$. For every $x > 0, \underline{f}(x) < \overline{f}(x)$ and for any $\upsilon > 1$,

$$\Pr\{f(x, r_t) \leq \upsilon \underline{f}(x)\} > 0, \Pr\{f(x, r_t) \geq \frac{1}{\upsilon}\overline{f}(x)\} > 0.$$

Assumption (**T.5**) ensures that the distribution of output from any current investment is non-degenerate and that $[\underline{f}(x), \overline{f}(x)]$ is the (essential) support of this distribution. Note that continuity of $\overline{f}$ assumed in **(T.5)** implies that $\overline{f}(K) = K$. Under (**T.5**), the functions $H, \overline{H}$ and $\underline{H}$ defined in Section 2 are continuous and strictly increasing in y. Further, for all $y \in (0, K]$ and $r \in A$,

$$K \geq \overline{H}(y) \geq H(y, r) \geq \underline{H}(y) > 0.$$

As $x(y) > 0$ for all $y > 0$, using assumption (**T.4**) and (**T.5**), $\overline{H}(y) > \underline{H}(y)$ for all $y > 0$ and $\overline{H}(y) < y$, for all $y > K$. Define

$$\beta = \inf\{y > 0 : \overline{H}(y) \leq y\}. \tag{61}$$

Lemma 8 *Assume* ***(T.3)***, ***(T.4)***, ***(T.5)*** *and Condition* **R**. *Then,* $\beta \in (0, K]$ *and further, (i)* $\overline{H}(y) > y$, *for all* $y \in (0, \beta)$, *(ii)* $\overline{H}(\beta) = \beta$ *and (iii)* $\underline{H}(y) < y$ *for all*

$y \geq \beta$.

Proof. From Proposition 1, we have $\lim_{y\to 0}\inf \frac{x(y)}{y} \geq \theta$. Observe that $E(\ln \theta B(r)) \leq \ln E(\theta B(r))$ so that Condition **R** implies $\theta E(B(r)) > 1$. Choose $\lambda_0 \in (\frac{1}{\theta E(B(r))}, 1)$. There exists $\epsilon > 0$ such that for all $y \in (0, \epsilon)$ and $r \in A$,

$$\frac{H(y,r)}{y} = \frac{f(x(y),r)}{y} = \frac{f(x(y),r)}{x(y)}\frac{x(y)}{y} \geq B(r)\lambda_0\theta$$

so that $E(H(y,r)) \geq \lambda_0 E(B(r))\theta y > y$ and in particular, $\overline{H}(y) > y$, for all $y \in (0,\epsilon)$.Thus, $K \geq \beta > \epsilon > 0$ and $\overline{H}(\beta) = \beta$. This also implies that $\underline{H}(\beta) < \beta$. From the Ramsey-Euler equation (7):

$$\begin{aligned} u'(c(\beta)) &= \delta E[u'(c(H(\beta,r)))f'(x(\beta),r)] \\ &\geq \delta E[u'(c(\overline{H}(\beta)))f'(x(\beta),r)] = \delta E[u'(c(\beta))f'(x(\beta),r)] \end{aligned}$$

so that $\delta E[f'(x(\beta),r)] \leq 1$. As $x(y)$ is strictly increasing in y we have that for all $y > \beta, \delta E[f'(x(y),r)] < 1$. Once again using the Ramsey-Euler equation, for all $y > \beta$,

$$\begin{aligned} u'(c(y)) &= \delta E[u'(c(H(y,r)))f'(x(y),r)] \\ &\leq \delta u'(c(\underline{H}(y)))E[f'(x(y),r)] < u'(c(\underline{H}(y))) \end{aligned}$$

and as $c(y)$ is strictly increasing, this implies that $\underline{H}(y) < y$ for all $y > \beta$. This completes the proof. ■

We are now ready to state the result on global stability:

Proposition 7 *Assume **(T.3)**,**(T.4)**, **(T.5)** and Condition **R**. Then, there exists a globally stable invariant distribution for the stochastic process of optimal outputs $\{y_t(y)\}$ that assigns probability one to $(0,K]$. For any initial output $y \in (0,K]$, optimal outputs converge in distribution to this positive stochastic steady state.*

Proof. The proof is entirely based on Kamihigashi and Stachurski (2014), hereafter K-S. Let $S = (0,K]$ be the state space and let $\mathcal{F}$ be the set of all Borel subsets of S. Let Q be the associated kernel defined by:

$$Q(x,B) = \Pr\{H(x,r) \in B\}, \forall B \in \mathcal{F}.$$

Define the notion of stationary (invariant) distribution, unique stationary distribution and global stability of Q on S as in Section 2.1 in K-S. Note that global stability of Q on S is equivalent to the existence of a unique invariant distribution on S and convergence (in distribution) to this invariant distribution from all $y \in S$. Let P be the probability measure defined on the product space in the usual manner. From Theorem 1 in K-S, global stability of Q is established if: (a) Q is increasing, (b) Q has an excessive distribution, (c) Q is order reversing, and (d) Q is bounded in probability. These concepts are formally defined in Sections 2.1 and 2.2 of K-S.

From Remark 3 in K-S, it follows that since $H(y,r)$ is increasing in y, Q is increasing. As S has a greatest element (namely, K), Q has an excessive distribution (see Remark 2 in K-S). We now show that Q is order reversing. From Lemma 8, $\underline{H}(y) < y$ for all $y \in [\beta, K]$. As $\underline{H}$ is continuous, there exists $\widetilde{y} \in (0, \beta)$ such that $\underline{H}(y) < y$ for all $y \in [\widetilde{y}, K]$. It is sufficient to show that for any $y^1, y^2 \in (0, K], y^2 \geq y^1$ there exists $t \in \mathbb{N}_+$ such that

$$P\{y_t(y^2) \leq \widetilde{y}\} > 0 \text{ and } P\{y_t(y^1) \geq \widetilde{y}\} > 0.$$

As $\underline{H}(y) < y$ for all $y \in [\widetilde{y}, K]$. There exists $\tau_1 \in \mathbb{N}$ such that for any $t \geq \tau_1, \underline{H}^t(y^2) < \widetilde{y}$ (where $\underline{H}^i(.) = \underline{H}(\underline{H}^{i-1}(.)), \underline{H}^1 = \underline{H}$); using assumption (**T.5**),

$$P\{y_t(y^2) < \widetilde{y}\} > 0. \tag{62}$$

From Lemma 8, $\overline{H}(y) > y$ for all $y \in (0, \beta]$ and further, $\overline{H}(y) \geq \min\{y, \beta\}$ for all $y \in (0, K]$.As $\widetilde{y} \in (0, \beta)$, there exists $\tau_2 \in \mathbb{N}$ such that for any $t \geq \tau_2$, $\overline{H}^t(y^1) > \widetilde{y}$ (where $\overline{H}^i(.) = \overline{H}(\overline{H}^{i-1}(.)), \overline{H}^1 = \overline{H}$) and using assumption (**T.5)**,

$$P\{y_t(y^1) > \widetilde{y}\} > 0. \tag{63}$$

For $t \geq \max\{\tau_1, \tau_2\}$, both (62) and (63) hold and thus, Q is order reversing. Finally, we show that Q is bounded in probability i.e., the sequence $\{Q^t(y, .)\}$ is tight for all $y \in S$. Here, Q^t is the $t - th$ order kernel giving the probability of transiting from y to $B \in \mathcal{F}$ in t steps and formally defined by

$$Q^1 = Q, Q^t(y, B) = \int Q^{t-1}(z, B) Q(y, dz).$$

Now, for any $y \in S$, the sequence $\{Q^t(y, .)\}$ is tight if for any $\xi > 0$, there exists a compact set $D \subset S$ such that $Q^t(y, S - D) \leq \xi$ for all t. Proposition 2(i) shows that for any $y \in (0, K]$ and for any $\xi > 0$, there exists $\widehat{\alpha}(y) > 0$ such that

$$P\{y_t(y) < \widehat{\alpha}(y)\} < \xi \text{ for all } t.$$

Defining $D = [\ \widehat{\alpha}(y), K]$, we can see that Q is bounded in probability. The proof is complete. ■

References

[1] Benhabib, J., and A. Rustichini (1994): "A Note on a New Class of Solutions to Dynamic Programming Problems Arising in Economic Growth," *Journal of Economic Dynamics and Control,* 18, 807-813.

[2] Brock, W.A., and M. Majumdar (1978): "Global Asymptotic Stability Results for Multisector Models of Optimal Growth under Uncertainty where Future Utilities are Discounted," *Journal of Economic Theory,* 18, 225-43.

[3] Brock, W.A., and L.J. Mirman (1972): "Optimal Economic Growth and Uncertainty: the Discounted Case," *Journal of Economic Theory,* 4, 479-513.

[4] Chung, K.L. (1974): A Course in Probability Theory. Second Edition. Academic Press, New York.

[5] Clark, C.W. (2010): Mathematical Bioeconomics: The Mathematics of Conservation. Third Edition. Wiley.

[6] de Hek, P. (1999): "On Endogenous Growth under Uncertainty," *International Economic Review* 40, 727-44.

[7] de Hek, P., and S. Roy (2001): "On Sustained Growth under Uncertainty," *International Economic Review* 42, 801-14.

[8] Danthine, J.P., and J.B. Donaldson (1981): "Stochastic Properties of Fast vs. Slow Growing Economies," *Econometrica*, 49, 1007-33.

[9] Dudley, R.M. (2002): Real Analysis and Probability. Cambridge University Press, Cambridge.

[10] Hardy, G., J.E. Littlewood, and G. Pólya (1952): Inequalities. Second Edition. Cambridge University Press, Cambridge.

[11] Jones, L.E., R. Manuelli, H.E. Siu, and E. Stacchetti (2004): "Fluctuations in Convex Models of Endogenous Growth I: Growth Effects," *Review of Economic Dynamics,* 8, 780-804.

[12] Kamihigashi, T. (2006): "Almost Sure Convergence to Zero in Stochastic Growth Models," *Economic Theory,* 29, 231-37.

[13] Kamihigashi, T. (2007): "Stochastic Optimal Growth with Bounded or Unbounded Utility and with Bounded or Unbounded Shocks," *Journal of Mathematical Economics,* 43, 477-500.

[14] Kamihigashi, T., and J. Stachurski (2014): "Stochastic Stability in Monotone Economies," *Theoretical Economics* 9, 383-407.

[15] Levhari, D., and T.N. Srinivasan (1969): "Optimal Savings under Uncertainty," *Review of Economic Studies,* 36, 153-63.

[16] Majumdar, M., T. Mitra, and Y. Nyarko (1989): Dynamic Optimization under Uncertainty: Non-convex Feasible Set. In G. R. Feiwel (Ed.), Joan Robinson and Modern Economic Theory, MacMillan, New York, 545-90.

[17] Majumdar, M., and I. Zilcha (1987): "Optimal Growth in a Stochastic Environment: Some Sensitivity and Turnpike Results," *Journal of Economic Theory,* 43, 116–33

[18] Mirman, L.J., and I. Zilcha (1975): "On Optimal Growth under Uncertainty," *Journal of Economic Theory,* 11, 329-339.

[19] Mirman, L.J., and I. Zilcha (1976): "Unbounded Shadow Prices for Optimal Stochastic Growth Models with Uncertain Technology," *International Economic Review,* 17, 121-32.

[20] Mitra, T., and S. Roy (2006): "Optimal Exploitation of Renewable Resources under Uncertainty and the Extinction of Species," *Economic Theory,* 28,1-23.

[21] Mitra, T., and S. Roy (2012): "Sustained Positive Consumption in a Model of Stochastic Growth: The Role of Risk Aversion," *Journal of Economic Theory,* 147, 850-880.

[22] Mitra, T., and G. Sorger (2014): "Extinction in Common Property Resource Models: An Analytically Tractable Example," *Economic Theory,* 57, 41-57.

[23] Nishimura, K., and J. Stachurski (2005): "Stability of Stochastic Optimal Growth Models: A New Approach," *Journal of Economic Theory,* 122, 100-118.

[24] Nishimura, K., R. Rudnicki, and J. Stachurski (2006): "Stochastic Optimal Growth with Nonconvexities," *Journal of Mathematical Economics*, 42, 74-96.

[25] Olson, L.J. (1989): "Stochastic Growth with Irreversible Investment," *Journal of Economic Theory,* 47, 101-129.

[26] Olson, L.J., and S. Roy (2000): "Dynamic Efficiency of Renewable Resource Conservation under Uncertainty," *Journal of Economic Theory,* 95, 186-214.

[27] Olson, L.J., and S. Roy (2006): Theory of Stochastic Optimal Growth. In R.-A. Dana, C. Le Van, T. Mitra and K . Nishimura (Eds.), Handbook on Optimal Growth 1 Discrete Time, Springer, 297-335.

[28] Royden, H.L. (1988): Real Analysis. Third Edition. Macmillan.

[29] Schäl, M. (1975): "Conditions for Optimality in Dynamic Programming and for the Limit of n-Stage Optimal Policies to be Optimal," *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete* 32, 179–196.

[30] Stachurski, J. (2002): "Stochastic Optimal Growth with Unbounded Shock," *Journal of Economic Theory,* 106, 40-65.

[31] Stokey, N.L., and R.E. Lucas Jr. (1989): Recursive Methods in Economic Dynamics. Cambridge, MA: Harvard University Press.